02-01-2020

MEAN VALUE THEOREM & MULTIVARIABLE CALCULUS

ROLLE'S THEOREM:

St: Suppose a, b ($a<b$) are two real constants. If $f:[a\ b] \to R$ be a function satisfying the following conditions.

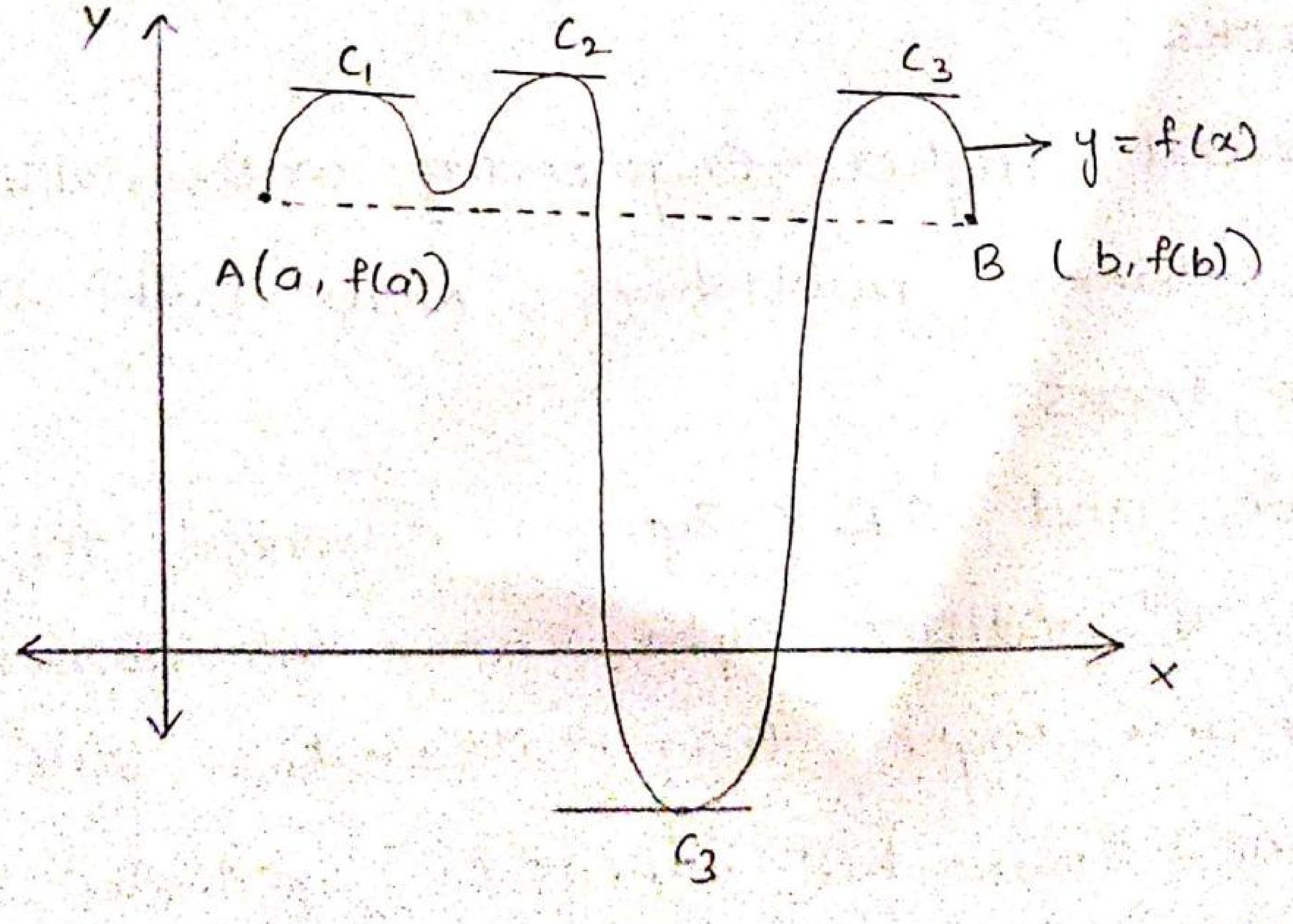
(i) 'f' is continous on $[a\ b]$.

(ii) 'f' is differentiable on (a, b)

(iii) $f(a) = f(b)$

Then $\exists$ atleast one 'c' $\in (a, b)$ such that $f'(c)=0$.

Geometrically there exists atleast one point 'c' on the curve $y=f(x)$ where the tangent is parallel to x-axis.



1) Verify Rolle's theorem for $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a, b]$, where $0<a<b$ i.e., $a>0$ & $b>0$.

Sol: Given $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$

Here, $f(x) = \log(x^2+ab) - \log(x) - \log(a+b)$

$$f'(x) = \frac{1}{x^2+ab}\cdot(2x) - \frac{1}{x} - 0.$$

$$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}, \quad \forall\, x \in (a,b)$$

$f(x)$ is continous on $[a,b]$ and differentiable on (a,b)

Now, $f(a) = \log\left(\frac{a^2+ab}{a(a+b)}\right) = \log(1) = 0.$

$f(b) = \log\left(\frac{b^2+ab}{b(a+b)}\right) = \log(1) = 0.$

$\therefore f(a) = f(b).$

$\therefore$ All conditions satisfied.

So, $\exists$ atleast one 'c' $\in (a,b)$ i.e., $a < c < b$ such that $f'(c) = 0$.

i.e., $\frac{2c}{c^2+ab} - \frac{1}{c} = 0$

$$\frac{2c}{c^2+ab} = \frac{1}{c}$$

$$2c^2 = c^2+ab$$

$$c^2 = ab$$

$$c = \pm\sqrt{ab} \in (a,b).$$

Hence verified.

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(2) Verify Rolle's theorem for $f(x) = \frac{\sin x}{e^x}$ in $[0,\pi]$

Sol: Given, $f(x) = \frac{\sin x}{e^x}$ on $[0, \pi]$

$$f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}}$$

$$f'(x) = \frac{e^x (\cos x - \sin x)}{(e^x)^{\cancel{2}}}$$

$$f'(x) = \frac{\cos x - \sin x}{e^x}, \quad \forall \; x \in (0, \pi)$$

Here, $f(x)$ is continous on $[0,\pi]$ and differentiable on $(0,\pi)$.

Now, $f(a) = \frac{\sin 0}{e^0} = 0$, $f(b) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$.

Hence, $f(a) = f(b)$ i.e., $f(0) = f(\pi)$.

$\therefore$ 3 conditions satisfied.

Then $\exists$ atleast one $c \in (0,\pi)$ such that $f'(c) = 0$.

i.e., $f(c) = \frac{\sin c}{e^c}$, $f'(c) = \frac{\cos c - \sin c}{e^c} = 0$

$$\cos c - \sin c = 0$$

$$\sin c = \cos c$$

$$\tan c = 1$$

$$c = \tan^{-1}(1)$$

Here $c = \tan^{-1}(1) = \frac{\pi}{4} \in (0,\pi)$

Hence verified.

8) Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in $[-3,0]$

Sol: Given, $f(x) = (x^2+3x)\, e^{-x/2}$

$$f'(x) = (2x+3)\, e^{-x/2} - \frac{1}{2}(x^2+3x)\, e^{-x/2}$$

$$f'(x) = e^{-x/2}\left(2x + 3 - \frac{(x^2+3x)}{2}\right)$$

$$f'(x) = e^{-x/2}\left(\frac{-x^2+x+6}{2}\right)$$

$\therefore$ $f(x)$ is continous on $[-3,0]$ and differentiable on $(-3,0)$.

Here $f(-3) = 0$, $f(0)=0$.

$\therefore$ $f(-3) = f(0)$

$\therefore$ 3 conditions satisfied.

So, $\exists$ atleast one 'c' $\in$ $(-3,0)$ such that $f'(c)=0$.

i.e., $e^{-c/2}\frac{(-c^2+c+6)}{2} = 0$

$$c^2-c-6 = 0$$

$$c = -2, 3$$

Here $c = -2 \in (-3,0)$.

Hence verified.

4) Verify Rolle's theorem for $f(x) = \tan x$ in $[0,\pi]$.

Sol: Rolle's theorem is not applicable for $f(x) = \tan x$ in $[0,\pi]$. Since, $f(x)$ is not continous at $x = \frac{\pi}{2}$

5) Verify Rolle's theorem for $f(x) = |x|$ in $[-1,1]$

Sol: Clearly $f(x)$ is continous on $[-1,1]$, but on differentiating $f(x)$ at $x=0$.

Now, $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

$$L.H.D = f'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{-x-0}{x-0}$$

$$= \underset{x\to 0}{Lt} \frac{-x}{x}$$

$$= -1$$

$$R.H.D = f'(0) = \underset{x\to 0+}{Lt} \frac{f(x)-f(0)}{x-0} = \underset{x\to 0}{Lt} \frac{x-0}{x-0} = 1$$

$\therefore$ L.H.D $\neq$ R.H.D.

$\therefore$ $f(x)$ is not differentiable.

Hence Rolle's theorem not applicable.

6) Verify Rolle's theorem for $f(x) = x^3$ in $[1,3]$.

Sol: Rolle's theorem is not applicable for the given function. since $f(1) \neq f(3)$.

7) Verify Rolle's theorem for $f(x) = (x-a)^m . (x-b)^n$ in $[a,b]$ where m,n are positive integers.

Sol: Given. $f(x) = (x-a)^m (x-b)^n$ on $[a,b]$.

$$f'(x) = (x-a)^m . n(x-b)^{n-1} + (x-b)^n . m(x-a)^{m-1}$$

$$f'(x) = (x-a)^{m-1}(x-b)^{n-1}\left[m(x-b)+n(x-a)\right]$$

$$\therefore f'(x) = (x-a)^{m-1} . (x-b)^{n-1}\left[x(m+n) - (na+mb)\right],$$

$$\forall\ x \in (a,b)$$

$\therefore$ $f(x)$ is continous on $[a,b]$ and differentiable on (a,b).

Also, $f(a) = 0$, $f(b) = 0$ so $f(a) = f(b)$.

i.e., $(c-a)^{m-1} . (c-b)^{n-1}\left[c(m+n) - (na+mb)\right] = 0$

$$c(m+n) - (na+mb) = 0$$

$$c(m+n) = na+mb$$

$$c = \frac{na+mb}{m+n} \in (a,b)$$

Hence verified.

8) Verify Rolle's theorem for $f(x) = e^x(\sin x - \cos x)$ in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

Sol: $f(x) = e^x(\sin x - \cos x)$

$f'(x) = e^x(\cos x + \sin x) + (\sin x - \cos x)e^x$.

$f'(x) = e^x(\cancel{\cos x} + \sin x + \sin x - \cancel{\cos x})$

$f'(x) = 2e^x \sin x$.

∴ It is continous on $[a,b]$ and differentiable on (a,b).

Now, $f(a) = f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = 0$.

$f(b) = f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = 0$.

∴ $f(a) = f(b)$. i.e., $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right)$

Then $\exists$ atleast one value 'c' $\in (a,b)$ i.e., $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

i.e., $f'(c) = 0$

$2e^c \sin c = 0$

$\sin c = 0$

$\boxed{c = \pi} \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Hence veified.

9) Verify Rolle's theorem for $f(x) = \frac{1}{x^2}$ in $[-1, 1]$.

Sol: $f(x)$ is not continous at $x = 0$.

Hence, Rolle's theorem is not applicable.

Lagrange's Mean Value Theorem (LMVT):

St: Suppose a, b ($a<b$) are two real numbers. Let $f:[a,b] \rightarrow R$ be a function satisfying the following conditions.

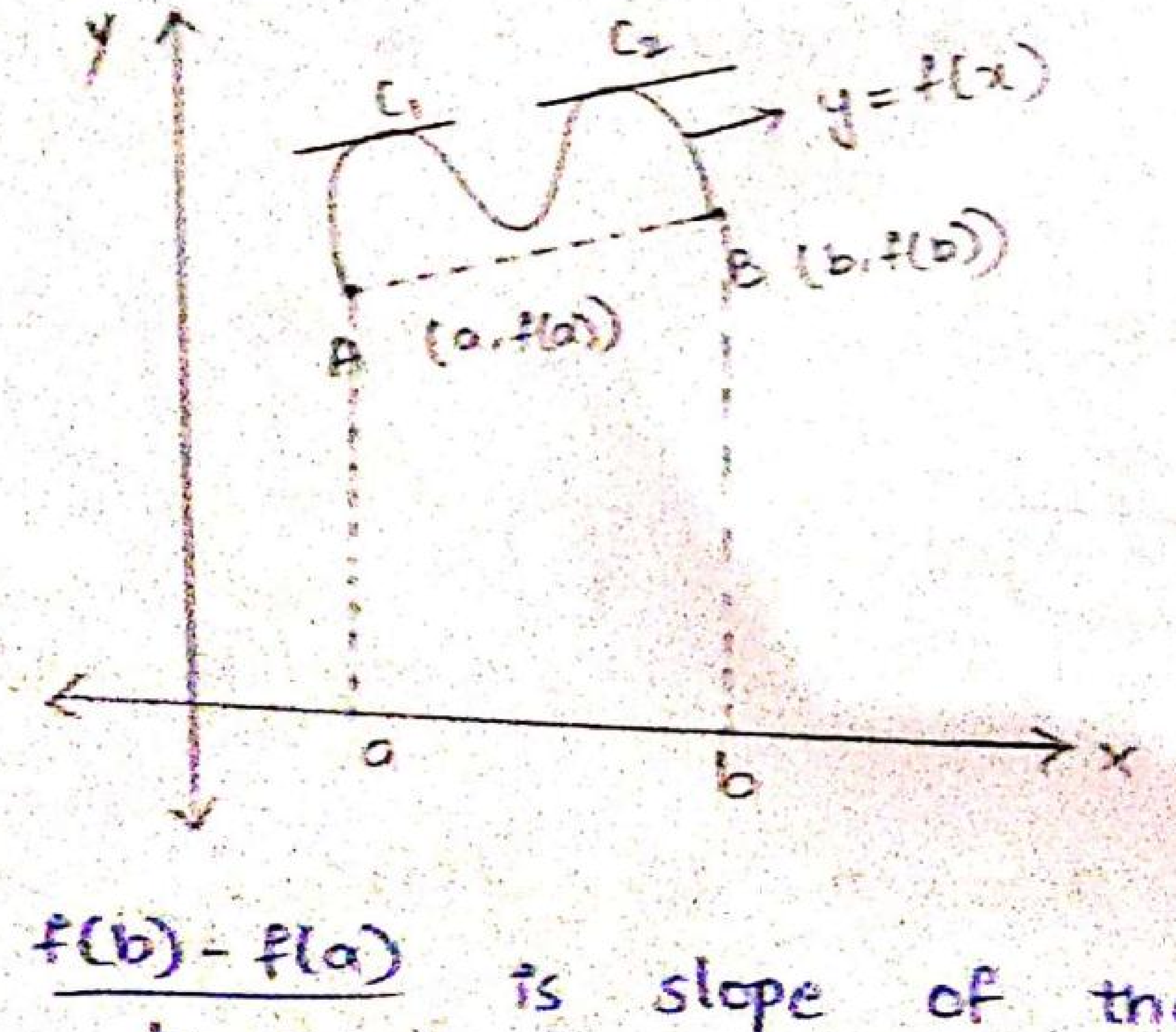
(i) 'f' is continous on $[a,b]$.

(ii) 'f' is differentiable on (a,b)

Then, $\exists$ atleast one 'c' $\in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Suppose $f(x)$ be a function such that it is continous on $[a,b]$ and differentiable on (a,b) then there exists atleast one point on the curve where the tangent is parallel to chord joining points A & B.



Here $\frac{f(b)-f(a)}{b-a}$ is slope of the chord joining points 'A' and 'B'.

1) Verify Lagrange's mean value theorem for

$f(x) = (x-1)(x-2)(x-3)$ in $[0,4]$.

Sol: Given, $f(x) = (x^2-3x+2)(x-3)$

$f(x) = x^3-6x^2+11x-6$

$f'(x) = 3x^2-12x+11$.

$\therefore$ $f(x)$ is continous on $[0,4]$ and differentiable on $(0,4)$.

By lagarange's theorem, $\exists$ atleast one $c \in (0,4)$

where $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

i.e., $3c^2-12c+11 = \dfrac{f(4)-f(0)}{4-0}$

$f'(c) = \dfrac{6+6}{4} = \dfrac{12}{4} = 3$

i.e., $3c^2-12c+11 = 3$

$3c^2-12c+8 = 0$

$c = \dfrac{12 \pm 4\sqrt{3}}{6}$

$c = \dfrac{2 \pm 2\sqrt{3}}{3} \in (0,4)$.

Hence verified.

2) Verify lagrange's mean value theorem for

$f(x) = \log_e x$ in $[1,e]$.

Sol: $f'(x) = \frac{1}{x}$, $\forall$ $x \in (1,e)$

$f(x)$ is continous on $[1,e]$ and differentiable on $(1,e)$

By LMVT, $\exists$ atleast one $c \in (1, e)$, such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1}$$

$$\frac{1}{c} = \frac{1-0}{e-1}$$

$$e - 1 = c$$

$$c = e - 1 \in (1, e)$$

Hence verified.

(3) Using lagrange's mean value theorem, prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ and hence deduce that $\frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$ where $0 < a < b$ i.e., $a > b$.

Sol: Let $f(x) = \tan^{-1}x$ in $[a,b]$ where $0 < a < b$.

$$f'(x) = \frac{1}{1+x^2}, \forall x \in (a,b)$$

Here $f(x)$ is continous on $[a,b]$ and differentiable on (a,b).

By LMVT, $\exists$ atleast one $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b - a} \rightarrow (1)$$

We know that $c \in (a, b)$

i.e., $a < c < b$

$$a^2 < c^2 < b^2$$

$1+a^2 < 1+c^2 < 1+b^2$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

from ① $\Rightarrow \frac{1}{1+a^2} > \frac{\tan^{-1}b - \tan^{-1}a}{b-a} > \frac{1}{1+b^2}$

$$\frac{b-a}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{\cancel{b-a}} < \frac{b-a}{1+a^2} \rightarrow ②$$

<u>Deductions</u>: put $a=1$, $b=\frac{4}{3}$ in ②

② $\Rightarrow \frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+(1)^2}$

$$\frac{1/3}{25/9} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1/3}{2}$$

$$\frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6} \quad //$$

④ Prove that $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3} - \frac{1}{8}$, by using lagranges mean value theorem.

<u>Sol</u>: Let $f(x) = \cos^{-1}x$ in $[a,b]$, where $0 < a < b$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}, \quad \forall x \in (a,b)$$

$\therefore f(x)$ is continous on $[a,b]$ and differentiable on (a,b)

By LMTVT, $\exists$ atleast one $c \in (a,b)$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

i.e., $\frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1}b - \cos^{-1}a}{b-a} \rightarrow ①$

we know that $c \in (a,b)$

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i.e., $a < c < b$

$a^2 < c^2 < b^2$

$-a^2 > -c^2 > -b^2$

$1-a^2 > 1-c^2 > 1-b^2$

$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{\cos^{-1}b - \cos^{-1}a}{b-a} > \frac{-1}{\sqrt{1-b^2}} \quad (\because \text{from } ①)$$

$$\frac{a-b}{\sqrt{1-a^2}} > \cos^{-1}b - \cos^{-1}a > \frac{a-b}{\sqrt{1-b^2}} \longrightarrow ②$$

Deduction: put $a = \frac{1}{2}$, $b = \frac{3}{5}$ in ②

$$② \Rightarrow \frac{\frac{1}{2} - \frac{3}{5}}{\sqrt{1-(\frac{1}{2})^2}} > \cos^{-1}(3/5) - \cos^{-1}(1/2) > \frac{\frac{1}{2} - \frac{3}{5}}{\sqrt{1-(\frac{3}{5})^2}}$$

$$\frac{-1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{3} > -\frac{1}{8}$$

$$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3} - \frac{1}{8}$$

Hence proved.

5) If $f(x) = \sin^{-1}x$, $0 < a < b < 1$, Use MVT to prove

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}.$$

Sol: Given $f(x) = \sin^{-1}x$, $[a\ b]$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}, \ \forall\ x \in (a,b)$$

If 'f' is continous on $[a\ b]$ and differentiable on (a,b)

By LMVT, $\exists$ atleast one $c \in (a,b)$, such that

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$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}b - \sin^{-1}a}{b - a} \rightarrow (1)$$

we know, $c \in (a,b)$ i.e., $a < c < b$

$$a^2 < c^2 < b^2$$

$$-a^2 > -c^2 > -b^2$$

$$1-a^2 > 1-c^2 > 1-b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

i.e.,

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}b - \sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}} \quad (\because \text{from } (1))$$

$$\frac{b-a}{\sqrt{1-a^2}} < \frac{\sin^{-1}b - \sin^{-1}a}{b-a} < \frac{b-a}{\sqrt{1-b^2}}.$$

Hence proved.

6) Prove that $\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}$, $0 < a < b$, hence show that $\frac{1}{4} < \log(4/3) < \frac{1}{3}$

Sol: Let $f(x) = \log_e x$ in $[a,b]$, where $0 < a < b$.

$f'(x) = \frac{1}{x}$, $\forall x \in (a,b)$

$\therefore f(x)$ is continous on $[a,b]$ & differentiable on (a,b).

By LMVT, $\exists$ atleast one $c \in (a,b)$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

i.e.,
$$\frac{1}{c} = \frac{\log_e b - \log_e a}{b-a}$$

$$\frac{1}{c} = \frac{\log_e\left(\frac{b}{a}\right)}{b-a} \longrightarrow (1)$$

we know, $a < c < b$ $\quad (\because c \in (a,b))$

$$\frac{1}{a} > \frac{1}{c} > \frac{1}{b}$$

$$\frac{1}{a} > \frac{\log_e\left(\frac{b}{a}\right)}{b-a} > \frac{1}{b}$$

$$\frac{b-a}{b} < \log_e\left(\frac{b}{a}\right) < \frac{b-a}{a} \rightarrow (2)$$

Deduction:

from (2), put $a = 3$, $b = 4$

$$(2) \Rightarrow \frac{4-3}{4} < \log\left(\frac{4}{3}\right) < \frac{4-3}{3}$$

$$\frac{1}{4} < \log\left(\frac{4}{3}\right) < \frac{1}{3}$$

Hence proved.

CAUCHY'S MEAN VALUE THEOREM:

Suppose $f: [a,b] \rightarrow R$, $g: [a,b] \rightarrow R$ be the functions, such that

1) $f(x)$, $g(x)$ are continous on $[a,b]$
2) $f(x)$, $g(x)$ are differentiable on (a,b)
3) $g'(c) \neq 0$

4) $g(a) \neq g(b)$

Then $\exists$ atleast one value 'c' $\in (a,b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

1) If $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$, prove that 'c' of cauchy's mean value therem is the geometric mean of a, b

(or) Verify cauchy's mean value theorem for $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$

Sol: Let $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$, where $0<a<b$

$f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = \frac{-1}{2x^{3/2}}$ $\left(\because \frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}\right)$

$g'(x) = \frac{-1}{2x\sqrt{x}}$, $\forall x \in (a,b)$

$\therefore f(x), g(x)$ are continous on $[a,b]$ and differentiable on (a,b).

By Now, $g'(c) \neq 0$ and $g(a) \neq g(b)$

By CMVT, $\exists$ atleast one $c \in (a,b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\frac{1}{\cancel{2}\cancel{\sqrt{c}}}}{\frac{-1}{\cancel{2}c\cancel{\sqrt{c}}}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$-c = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}$$

$$\cancel{+}c = \sqrt{ab} \times \cancel{+}\frac{\cancel{(\sqrt{b}-\sqrt{a})}}{\cancel{(\sqrt{b}-\sqrt{a})}}$$

$\boxed{c = \sqrt{ab}}$ i.e., $c^2 = ab$ (a, c, b)

Thus, 'c' of Cauchy's mean value theorem is geometric mean of a, b

2) Verify Cauchy's mean value theorem for $f(x) = e^x$, $g(x) = e^{-x}$ in $[a,b]$.

Sol: Given, $f(x) = e^x$, $g(x) = e^{-x}$, $[a,b]$

$f'(x) = e^x$, $g'(x) = -e^{-x}$, $\forall (a,b)$

$f(x)$, $g(x)$ are continous on $[a,b]$ and differentiable on (a,b)

& Here $g'(c) = -e^{-c} \neq 0$

Here $g(a) \neq g(b)$

i.e., $e^{-a} \neq -e^{-b}$

Hence verified.

$\exists$ atleast one 'c' $\in (a,b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$\frac{-e^c}{1/e^c} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$-e^{2c} = \frac{e^b - e^a}{\frac{e^a - e^b}{e^b e^a}}$$

$$+(e^c)^2 = e^{b+a} \cdot \frac{\cancel{(e^b - e^a)}}{+\cancel{(e^b - e^a)}}$$

$$e^{2c} = e^{a+b}$$

Since, bases are equal we can equate the powers

$$2c = a + b$$

$$c = \frac{a+b}{2} \in (a,b)$$

Hence verified

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3) Verify CMVT for $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in $[a,b]$ for $[a$ where $0 < a < b$ $\quad c = \frac{2ab}{a+b}$

4) Verify CMVT for $f(x) = \sin x$, $g(x) = \cos x$ in $[0, \frac{\pi}{2}]$. $c = \frac{\pi}{4}$

3) Given, $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ $\quad [a,b]$

$$f'(x) = \frac{-2}{x^3}, \quad g'(x) = \frac{-1}{x^2}, \quad \forall\ x \in (a,b)$$

$f(x)$, $g(x)$ are continous on $[a,b]$ and differentiable on (a,b)

Here $g'(c) = \frac{-1}{c^2} \neq 0$

Here $g(a) \neq g(b)$

$$\frac{1}{c} \neq -\frac{1}{c^2}$$

$\exists$ atleast one 'c' $\in (a,b)$, such that.

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{+2/\cancel{x}c^3}{+1/c^2} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$$

$$\frac{2}{c} = \frac{a^2 - b^2/(ab)^2}{(a-b)/(ab)}$$

$$\frac{2}{c} = \frac{(a+b)\cancel{(a-b)}}{\cancel{(a-b)}(ab)}$$

$$c = \frac{2ab}{a+b} \in (a,b)$$

Hence verified.

4) Given, $f(x) = \sin x$, $g(x) = \cos x$ $\quad [0, \frac{\pi}{2}]$

$$f'(x) = \cos x, \quad g'(x) = -\sin x, \quad \forall\ x \in (0, \tfrac{\pi}{2})$$

$f(x)$, $g(x)$ are continous on $[0, \frac{\pi}{2}]$ and differentiable on $(0, \frac{\pi}{2})$.

Here $g'(c) = -\sin c \neq 0$.

Here $g(a) \neq g(b)$ i.e., $\frac{1}{a} \neq \frac{1}{b}$ $\cos a \neq \cos b$

$\exists$ atleast one 'c' $\in (a,b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\cos c}{-\sin c} = \cancel{\frac{\frac{1}{b^2}-\frac{1}{a^2}}{}} \quad \frac{\sin b - \sin a}{\cos b - \cos a}$$

$$\frac{\cos c}{-\sin c} = \frac{\sin\left(\frac{\pi}{2}\right) - \sin 0}{\cos\left(\frac{\pi}{2}\right) - \cos 0}$$

$$-\cot c = \frac{1-0}{0-1}$$

$$+\cot c = +1$$

$$c = \cot^{-1}(1)$$

$$c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence verified.

5) Verify cauchy's mean value theorem for $f(x)$ and $f'(x)$ where $f(x) = \log_e x$ and $f'(x) = \frac{1}{x}$ in $(1, e)$.

Sol: Given, $f(x) = \log_e x$, let $g(x) = \frac{1}{x} = f'(x)$

$$f'(x) = \frac{1}{x} \qquad g'(x) = \frac{-1}{x^2} \cdot \forall x \in (1,e)$$

$\therefore$ $f(x)$, $g(x)$ are continous on $(1,e)$ and differentiable on $(1,e)$ also $g'(c) = \frac{-1}{c^2} \neq 0$ and

$$g(a) \neq g(b) \text{ i.e., } \frac{+1}{a^2} \neq \frac{+1}{b^2}$$

$\therefore$ By CMVT, $\exists$ atleast one 'c' $\in (1,e)$ Such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{1/c}{-1/c^2} = \frac{\log_e e - \log_e 1}{\frac{1}{e} - \frac{1}{1}}$$

$$-c = \frac{1-0}{1-e/e}$$

$$-c = \frac{e}{1-e}$$

$$c = \frac{e}{e-1} \in (1, e)$$

Hence verified.

⇒ Taylor's and maclaurin's series :

The Taylor series expansion of $f(x)$ in powers of $x-a$ or about $x=a$ is given by

$$f(x) = f(a) + \frac{(x-a)^1}{1!} \cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$$\cdots + \frac{(x-a)^n}{n!} f^{(n)}a + \cdots \longrightarrow (1)$$

put $a=0$ in equ (1), we get

$$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots +$$

$$\frac{x^n}{n!} f^{(n)}(0) + \cdots$$

which is known as maclaurin's series of $f(x)$.

I) Using maclaurin's series expand the following.

1) $\cos x$, 2) $\sinh x$, 3) $\tan^{-1} x$

1) $\cos x$

Sol: Let $f(x) = \cos x$

By maclaurin's series $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots \longrightarrow (1)$

Now, $f(0) = \cos 0 = 1$

$f'(x) = -\sin x$

$f'(0) = -\sin 0 = 0$

$f''(0) = -\cos x = -1$

$f'''(0) = \sin x = 0$

$f^{IV}(0) = \cos x, = 1$

Substituting the above values in ①, we get

$$\cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) + \ldots$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

2) $\sinh x$.

Sol: Let $f(x) = \sinh x$

By maclaurin's series $f(x) = f(0) + \frac{x^1}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots \rightarrow$ ①

Now, $f(0) = \sinh(0) = 0$, $f'(0) = \cosh x$

$f'(0) = \cosh(0) = 1$

$f''(0) = +\sinh(0) = 0$

$f'''(0) = +\cosh(0) = 1$

$f^{IV}(0) = \sinh(0) = 0$

Substituting the above values in ①, we get

$$\sinh x = 0 + \frac{x^1}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(0) + \ldots$$

$$\therefore \sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$

3) $\tan^{-1}x$

Sol: Let $f(x) = \tan^{-1}x$

By maclaurin's series,

$$f(x) = f(0) + \frac{x^1}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots \rightarrow ①$$

Now, $f(0) = \tan^{-1}(0) = 0$, $f'(x) = \frac{1}{1+x^2}$

$\boxed{f'(0) = \frac{1}{1+0} = 1}$

$f''(0) = \frac{-1}{(1+x^2)^2} \cdot 2x \cdot = 0$

$f''(0) = 0$

Consider, $f'(x) = \frac{1}{1+x^2} \rightarrow$ (I)

from (I), $f'(x)(1+x^2) = 1 \rightarrow$ (2)

Differentiating (2) with respect to 'x'.

$f''(x).(1+x^2) + f'(x)(2x) = 0 \rightarrow$ (3)

$f''(0)(1) + 0 = 0$

$\boxed{f''(0)=0}$

Differentiate equ (3) with respect to 'x' we get

$f'''(x)(1+x^2) + f''(x)(2x) + f''(x).2x + 2\cancel{x}f'(x) = 0 \rightarrow$ (4)

$f'''(0)(1+0) + f''(0)2(0) + f''(0).2(0) + 2f'(0) = 0$

$f'''(0) + 0 + 0 + 2(1) = 0$

$\boxed{f'''(0) = -2}$

Differentiate (4) with respect to 'x'.

$f^{IV}(x).(1+x^2) + f'''(x).(2x) + 4\left[f'''(x)\,x + f''(x).1\right] + 2f''(x) = 0$

$f^{IV}(0)(1+0^2) + f'''(0)2(0) + 4\left[f'''(0).(0) + f''(0).1\right] + 2f''(0) = 0$

$f^{IV}(0) + 0 + 0 = 0$

$\boxed{f^{IV}(0) = 0}$ Similarly, $\boxed{f^{V}(0) = 24}$

Substituting the above values in (1), we get.

$\tan^{-1}x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!} \times 24 + \cdots$

$\therefore \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(4) Obtain maclaurin's series for $e^{\sin x}$ upto $x^{IV\,th}$ term.

Sol: Let $f(x) = e^{\sin x}$.

By maclaurin's series,

$f(x) = f(0) + \frac{x^1}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \rightarrow$ (1)

Now, $f(0) = e^{\sin 0} = e^0 = 1$, $f'(x) = e^{\sin x}.(\cos x)$

$f'(0) = e^{\sin(0)}.\cos 0 = 1$

$f''(x) = e^{\sin x}(-\sin x) + \cos x . e^{\sin x}. \cos x$

$f''(0) = e^{\sin 0}(-\sin 0) + \cos 0 \, e^{\sin 0} . \cos 0$

$f''(0) = 1$

$f'''(x) = e^{\sin x}.(-\cos x) + (-\sin x). e^{\sin x}.\cos x + e^{\sin x}.\cos^2 x$

$+ e^{\sin x} . 2\cos x (-\sin x) + \cos^2 x . e^{\sin x}. \cos x$

$f'''(0) = e^{\sin 0}(-\cos 0) + (-\sin 0) e^{\sin 0}. \cos 0 + e^{\sin 0}. 2\cos 0 .$
$(-\sin 0) + \cos^2 0 . e^{\sin 0}.\cos 0$

$f'''(0) = -1 + 0 + 0 + 1$

$f'''(0) = 0$

$f^{IV}(x) = e^{\sin x}. (+\sin x) + (-\cos x) e^{\sin x}. (\cos x) +$

Similarly, $f^{IV}(0) = 0 + 0 + 2 \left[1(-1) + 1(0)\right] + 1 \times 0 - 1$

$= -2 - 1$

$= -3$

Sub these values in equ 1 , we get:

$e^{\sin x} = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-3) + \cdots$

$e^{\sin x} = 1 + \frac{x}{1} + \frac{x^2}{2} - \frac{x^4}{8}$

$f'''(x) = e^{\sin x}.(-\cos x) + e^{\sin x}. \sin x . \cos x + 2 e^{\sin x}. \sin x \cos x$

$+ e^{\sin x}. \cos^3 x .$

$f^{IV}(x) = e^{\sin x}.(\sin x) + (-\cos x) e^{\sin x}. \cos x + \left[e^{\sin x} \sin x \frac{d}{dx}(-\sin x) \right.$
$\left. + e^{\sin x}. \cos x . \cos x + \sin x \cos x . e^{\sin x}.\cos x \right] - 2$

[2. $e^{\sin x}$

$+ e^{\sin}$

$f^{IV}(0) =$

$f^{IV}(0)$

$f^{IV}(0$

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Sub

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$\left[2.\ e^{\sin x}\sin x(-\sin x) + e^{\sin x}.\cos x.\cos x + \cos x \sin x . e^{\sin x}.\cos x\right]$

$+ e^{\sin x}.3\cos^2 x(-\sin x) + \cos^3 x . e^{\sin x}.\cos x\right]$

$f^{IV}(0) = e^{\sin 0}.\sin 0 - e^{\sin 0}.\cos^2 0 + e^{\sin 0}\sin^2 0 - e^{\sin 0}\cos^2 0$

$- e^{\sin 0}.\sin 0\cos^2 0\big] + 2e^{\sin 0}\sin^2 0 - 2e^{\sin 0}.\cos^2 0.$

$- 2e^{\sin 0}.\sin 0\cos^2 0 + e^{\sin 0}.3\cos^2 0(-\sin 0) + \cos^4 0\, e^{\sin 0}$

$f^{IV}(0) = 0 - 1 + 0 - 1 - 0 + 0 - 2 - 0 + 0 + 1$

$f^{IV}(0) = -4 + 1$

$f^{IV}(0) = -3$

Sub the above values in ①. , we get

$$e^{\sin x} = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-3)$$

$$e^{\sin x} = 1 + \frac{x}{1} + \frac{x^2}{2} - \frac{x^4}{8}$$

5) $\log(x+1)$, hence deduce that $\log\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

Sol: Given, let $f(x) = \log(x+1)$

By maclaurin's series.

$$f(x) = f(0) + \frac{x^1}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \rightarrow ①$$

$f(0) = \log(0+1) = 0$, $f'(x) = \frac{1}{x+1}$

$f'(0) = \frac{1}{0+1} = 1$

$f''(x) = \frac{-1}{(x+1)^2}$, $f''(0) = \frac{-1}{(0+1)^2} = \frac{-1}{1} = -1$

$f''(0) = -1$

$f'''(x) = \frac{+2(1)}{(x+1)^3}$

$f'''(0) = \frac{2(1)}{(0+1)^3} = 2$

$f'''(0) = 2$

$$f^{IV}(0) = \frac{-3(2)}{(0+1)^4} = -3(2)$$

$$f^{IV}(0) = -3(2) = -6$$

substitute these values in equ ①, we get

$$\log(x+1) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \dots$$

$$\log(x+1) = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \rightarrow ②$$

Now,

$\log(1-x)$, replace 'x' by '$-x$'

$$\log(1-x) = -x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \rightarrow ③$$

Now, ② − ③, we get

$$\log(1+x) - \log(1-x) = 2x + \frac{2x^3}{3} + 2\frac{x^5}{5} + 2\frac{x^7}{7} + \dots$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right)$$

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{7} + \frac{x^7}{7} + \dots$$

$$\log\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$$

Hence deduced

6) Prove that $\log(\sec x) = \frac{x^2}{2} + \frac{1}{12}\cdot x^4 + \frac{1}{45}\cdot x^6 + \dots$

Sol: Given, $f(x) = \log(\sec x)$, $f'(x) = \frac{1}{\cancel{\sec x}}\cdot(\cancel{\sec x}.\tan x)$

$f(0) = \log(\sec 0) = 0$ $\quad$ $f''(x) = \sec^2 x$

$f'(0) = \tan 0 = 0$ $\quad$ $f'''(x) = 2\sec x.\sec x\tan x$

$f''(0) = \sec^2 0 = 1$ $\quad$ $= 2\sec^2 x\tan x$

$f'''(0) = 2\sec^2 0.\tan 0 = 0$

By maclaurin's series,

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \rightarrow (1)$

$f^{IV}(x) = 2.(\sec^2 x . \sec^2 x + \tan x \, 2\sec^2 x . \tan x)$

$f^{IV}(x) = 2 \sec^4 x + 2.2 \sec^2 x \tan^2 x$

$f^{IV}(0) = 2(\sec^4 0) + 4\sec^2 0 \tan^2 0$

$f^{IV}(0) = 2$

Substituting these values in (1), we get

$\log(\sec x) = 0 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(2) + \cdots$

$\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \cdots$

Hence proved.

7) Show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$

Hence show that $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$

Sol: Given, $f(x) = \log(1+e^x)$

By maclaurin's series,

$f(x) = f(0) + \frac{x'}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \rightarrow (1)$

$f(0) = \log(1+e^0) = \log 2$

$f'(x) = \frac{1}{1+e^x} \cdot e^x = \frac{e^x}{1+e^x}$; $f'(0) = \frac{e^0}{1+e^0} = \frac{1}{2}$

$\cancel{f''(x)} = \cancel{\log(1+e^x)} = \cancel{\log(1+e^0)} - \log$

$f'(x) = \frac{e^x}{1+e^x}$

$f'(x)(1+e^x) = e^x \rightarrow (2)$

Differentiate (2) w.r.t. 'x'.

$f''(x)(1+e^x) + f'(x) . e^x = e^x \rightarrow (3)$

$f''(0)(1+e^0) + f'(0) . e^0 = e^0$

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$f''(0)(1+1) + \frac{1}{2} = 1$

$f''(0)(2) = +\frac{1}{2}$

$f''(0) = \frac{1}{4}$

Differentiate (3) w.r.t. 'x', we get

$f''(x)\cdot e^x + (e^x+1) f'''(x) + f'(x)\cdot e^x + e^x\cdot f''(x) = e^x \rightarrow (4)$

$f''(0)\cdot e^0 + (e^0+1) f'''(0) + f'(0)\, e^0 + e^0\cdot f''(0) = e^0$

$\frac{1}{4}\cdot 1 + (1+1) f'''(0) + \frac{1}{2}\cdot 1 + 1\cdot \frac{1}{4} = 1$

$2f'''(0) + \cancel{2}\left(\frac{1}{\cancel{4}_2}\right) + \frac{1}{2} = 1$

$2f'''(0) + \cancel{1} = \cancel{1}$

$f'''(0) = 0$

Differentiate (4) w.r.t. 'x', we get.

$f''(x)\cdot e^x + e^x\cdot f'''(x) + (e^x+1) f^{IV}(x) + f'''(x)\cdot e^x + f'(x)\cdot e^x +$

$e^x f''(x) + e^x\cdot f'''(x) + f''(x) \cdot e^x = e^x$

$f''(0)\cdot e^0 + e^0\cdot f'''(0) + (e^0+1) f^{IV}(0) + f'''(0)\cdot e^0 + f'(0)\cdot e^0 +$

$e^0 f''(0) + e^0 f'''(0) + f''(0)\cdot e^0 = e^0$

$\frac{1}{4}(1) + (1)(0) + (1+1) f^{IV}(0) + \cancel{f}(0)\cdot(1) + \frac{1}{2}(1) + 1\left(\frac{1}{4}\right) +$

$1(0) + \frac{1}{4}(1) = 1$

$\frac{1}{4} + 2f^{IV}(0) + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$

$2f^{IV}(0) + \frac{1+1+1+2}{4} = 1$

$2f^{IV}(0) = 1 - \frac{5}{4}$

$f^{IV}(0) = \frac{-1}{8}$ //

Substituting these values in ①, we get

$$\log(1+e^x) = \log 2 + \frac{x}{1!}\cdot\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(\frac{1}{4}\right) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}\left(\frac{1}{8}\right) + \dots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^4}{192} + \dots \rightarrow ⑤$$

Differentiate ⑤ w.r.t. 'x', we get

$$\frac{e^x}{1+e^x} = 0 + \frac{1}{2}(1) + \frac{1}{8}(2x) + \frac{1}{192}\cdot 4x^3 + \dots$$

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{1}{4}\cdot x - \frac{x^3}{48} + \dots //$$

Hence proved

8) Show that $e^x \cdot \cos x = 1 + x - 2\cdot\frac{x^3}{3!} - \frac{2^2}{4!}\cdot x^4 - \dots$

Sol: Given, $f(x) = e^x \cdot \cos x$.

9) $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$

10) Expand $\log_e x$ in powers of $(x-1)$ and hence evaluate $\log_e(1.1)$ correct to 4 decimal places.

Sol: Let $f(x) = \log_e x$

Using Taylor's series, expansion of $f(x)$ about $x=a$ or powers of $x-a$

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

Here $a=1$

$$f(x) = f(1) + \frac{(x-1)}{1!}f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \dots \rightarrow ①$$

$f(1) = \log_e 1 = 0$

$f'(x) = \frac{1}{x}$, $f'(1) = \frac{1}{1} = 1$

$f''(x) = -\frac{1}{x^2}$, $f''(1) = \frac{-1}{1^2} = -1$

$f'''(x) = \frac{+2}{x^2}$, $f'''(1) = \frac{+2}{1^3} = +2$

$f^{IV}(x) = \frac{-6}{x^4}$, $f^{IV}(1) = -6$

Substituting these values in ① we get.

$$\log_e^x = (x-1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \ldots$$

$$\log_e^x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

Now, $\log_e(1.1) = 0.0953$

11) Expand $\sin x$ in powers of $x - \frac{\pi}{4}$ hence find the value of $\sin 91$ correct to 4 decimal places.

Sol: Given, $f(x) = \sin x$

Using Taylors series, expandsion of $f(x)$ about $x=a$ or powers of $x-a$.

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

Here $a = \frac{\pi}{4}$

$$f(x) = f\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)}{1!}f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right) + \ldots \rightarrow ①$$

$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x$ $\quad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x$ $\quad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$

$f'''(x) = -\cos x$ $\quad f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$

$f^{IV}(x) = \sin x$ $\quad f^{IV}\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

substituting these values in ①, we get.

$$\sin x = \frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)}{1!}\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{3!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^4}{4!}\left(\frac{1}{\sqrt{2}}\right) + \ldots$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)^2}{2\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)^3}{6\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)^4}{24\sqrt{2}} + \cdots$$

Now, $\sin 91 = 0.9998$

12) Expand e^x in powers of $(x-1)$ using Taylor's series

13) Using maclaurins series, expand 'Tanx' upo the term containing x^5.

8) Given $e^x \cos x = f(x)$

By maclaurin's series,

$$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \rightarrow (1)$$

$f(0) = e^0 \cos 0 = 1$

$f'(x) = e^x(-\sin x) + \cos x e^x$, $f'(0) = e^0(-\sin 0) + \cos 0 e^0 = 1$

$f''(x) = e^x(-\cos x) + (-\sin x) e^x + \cos x e^x + e^x(-\sin x)$

$f''(x) = -\cancel{\cos x e^x} - \sin x e^x + \cancel{\cos x e^x} - \sin x e^x$

$f''(x) = -2\sin x e^x$.

$f''(0) = -2\sin(0).e^0$

$f''(0) = 0$

$f'''(x) = -2\left(\sin x e^x + e^x \cos x\right)$

$f'''(0) = -2\left(\sin(0).e^0 + e^0 \cos 0\right)$

$f'''(0) = -2$

$f^{IV}(x) = -2\left[\cancel{\sin x e^x} + e^x \cos x + \cancel{e^x(-\sin x)} + \cos x e^x\right)$

$f^{IV}(x) = -2\left(2e^x \cos x\right)$

$f^{IV}(x) = -4e^x \cos x$

$f^{IV}(0) = -4e^0 \cos 0$

$f^{IV}(0) = -4$

substituting these values in (1), we get.

$$e^x.\cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2}{4!}x^4 + \cdots$$

Hence proved

9) Given, $f(x) = \log(1+\sin x)$

By maclaurin's series,

$$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \rightarrow (1)$$

$$f(0) = \log(1+\sin 0) = 0$$

$$f'(x) = \frac{\cos x}{1+\sin x}, \quad f'(0) = \frac{\cos 0}{1+\sin 0} = 1$$

$$f'(x) = \frac{\cos x}{1+\sin x} \rightarrow (1)$$

$$f'(x)(1+\sin x) = \cos x \rightarrow (1)$$

Differentiate (1) w.r.t. 'x'.

$$f'(x)(\cos x) + (1+\sin x) f''(x) = -\sin x \rightarrow (2)$$

$$f'(0)\cos 0 + (1+\sin 0) f''(0) = -\sin 0$$

$$1(1) + f''(0) = 0$$

$$f''(0) = -1$$

Differentiate (2) w.r.t. 'x'.

$$f'(x)(-\sin x) + \cos x . f''(x) + (1+\sin x) f'''(x) + f''(x)(\cos x) = -\cos x$$

$$f'(0)(-\sin 0) + \cos 0\, f''(0) + (1+\sin 0) f'''(0) + f''(0)\cos(0) = -\cos 0$$

$$0 + 1(-1) + (1) f'''(0) + (-1)(1) = -1$$

$$-1 + f'''(0) - 1 = -1$$

$$f'''(0) = 2-1$$

$$f'''(0) = 1$$

Similarly, $f^{iv}(0) = -2$

substitute these values in (1), we get

$$\log(1+\sin x) = 0 + \frac{x^1}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(-2)$$

$$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$$

Hence proved

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12) Given, $f(x) = e^x$

Using Taylor's series expansion of $f(x)$ about $(x-a)$ or powers of $x-a$.

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Here $a = 1$

$$f(x) = f(1) + \frac{(x-1)}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots \rightarrow (1)$$

$f(1) = e^1 = e$

$f'(x) = e^x \quad , \quad f'(1) = e^1$

$f''(x) = e^x \quad , \quad f''(1) = e^1$

$f'''(x) = e^x \quad , \quad f'''(1) = e^1$

Sub these values in (1), we get.

$$e^x = e + \frac{(x-1)}{1!} e + \frac{(x-1)^2}{2!} e + \frac{(x-1)^3}{3!} \cdot e. \quad //$$

13) Given, $\tan x = f(x)$

By maclaurin's series,

$$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \rightarrow (1)$$

$f(0) = \tan 0 = 0$

$f'(x) = \sec^2 x \quad , \quad f'(0) = \sec^2 0 = 1$

$f''(x) = 2\sec x \cdot \sec x \tan x \quad , \quad f''(0) = 2\sec^2 0 \tan 0$

$= 2\sec^2 x \tan x \qquad f''(0) = 2(1)(0)$

$f''(0) = 0$

$f'''(x) = 2[\sec^2 x \sec^2 x + \tan x \cdot 2\sec x \cdot \sec x \tan x]$

$f'''(x) = 2[\sec^4 x + 2\sec^2 x \tan^2 x]$

$f'''(0) = 2[\sec^4 0 + 2\sec^2 0 \tan^2 0]$

$f'''(0) = 2$

$f^{IV}(x) = 2[4\sec^3 x \cdot \sec x \tan x + 2[\sec^2 x\, 2\tan x \sec^2 x + \tan^2 x \cdot 2\sec x \sec x \tan x]]$

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$f^{IV}(x) = 2\left[4\sec^4 x \tan x + 4\tan x \cdot \sec^4 x + 4\sec^2 x \cdot \tan^3 x\right]$

$f^{IV}(x) = 2\left[4\sec^4 x \tan x + 4\tan\right.$

$f^{IV}(0) = 2\left[4\sec^4 0 \tan 0 + 4\tan 0 \sec^4 0 + 4\sec^2 0 \tan^3 0\right]$

$f^{IV}(0) = 0$

$f^{V}(x) = 8\left[\cancel{\sec^4 x} \sec^2 x + \tan x \, 4\sec^3 x \sec x \tan x \; 4 \; \cancel{\sec^4 x} \sec^2 x\right.$

$+$

$f^{IV}(x) = 8\left[2\sec^4 x \tan x + \sec^2 x \tan^3 x\right]$

$$f^{V}(x) = 8\left[2\left[\sec^4 x \sec^2 x + \tan x \, 4\sec^3 x \sec x \tan x\right] + \sec^2 x \, 3\tan^2 x \sec^2 x + \tan^3 x \, 2\sec x \cdot \sec x \tan x\right]$$

$$f^{V}(x) = 8\left[2\sec^6 x + 8\sec^4 x \tan^2 x + 3\sec^4 x \tan^2 x + 2\sec^2 x \tan^4 x\right]$$

$$f^{V}(0) = 8\left[2\sec^6(0) + 8\sec^4(0)\tan^2(0) + 3\sec^4(0)\tan^2(0) + 2\sec^2(0)\tan^4(0)\right]$$

$f^{V}(0) = 8\left[2(1)\right]$

$f^{V}(0) = 16$

Substitute these values in (1), we get

$$\tan x = 0 + \frac{x^1}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(16) + \dots \rightarrow (2)$$

$$\tan x = x + \frac{x^2}{2}(0) + \frac{x^3}{3} + (0) + \frac{x^5 \cdot 16}{120} + \dots$$

$$\tan x = x + \frac{x^3}{3} + 4 \cdot \frac{x^5}{30} + \dots$$

⇒ FUNCTIONS OF SEVERAL VARIABLES: (11-01-2020)

* Jacobians:

If u, v are functions of two independent variables x, y then the determinant of $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v with respect to x, y. And it is written as $J\left(\frac{u,v}{x,y}\right)$ (or) $\frac{\partial(u,v)}{\partial(x,y)}$.

$$\therefore J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Similarly the Jacobian of u, v, w w.r.t. x, y, z is

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

PROPERTIES:

1) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$, then $JJ' = 1$.

2) If u, v are functions of r, s and r, s are functions of x, y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$.

1) If $x = r\cos\theta$, $y = r\sin\theta$, evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$, $\frac{\partial(r,\theta)}{\partial(x,y)}$ & prove that $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Sol: Given, $x = r\cos\theta \rightarrow (1)$, $y = r\sin\theta \rightarrow (2)$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r(1)$$

$$= r$$

Now, $(1)^2 + (2)^2 \Rightarrow x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$$

$$x^2+y^2 = r^2(1)$$

$$x^2+y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2} \rightarrow (3)$$ r is a function of x, y.

Now, $\frac{(2)}{(1)} \Rightarrow \frac{y}{x} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \tan\theta$

$$\frac{y}{x} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \rightarrow (4)$$ θ is a function of x, y.

For our convinience,

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

So, $\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cancel{2}x = \frac{x}{\sqrt{x^2+y^2}}$; $\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} 2y = \frac{y}{\sqrt{x^2+y^2}}$

$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$; $\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{y^2}{(x^2+y^2)(\sqrt{x^2+y^2})}$$

$$= \frac{(x^2+y^2)}{(x^2+y^2)\ (x^2+y^2)^{1/2}}$$

$$= \frac{1}{(x^2+y^2)^{1/2}}$$

$$= \frac{1}{(r^2)^{1/2}} \quad (\because \text{from } ③)$$

$$= \frac{1}{r}$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \times \frac{1}{r} = 1.$$

Hence proved.

If $x = e^u \sec v$, $y = e^u \tan v$, find $J = \frac{\partial(x,y)}{\partial(u,v)}$, $J' = \frac{\partial(u,v)}{\partial(x,y)}$. Hence show that $JJ' = 1$. $u = \frac{1}{2}\log(x^2-y^2)$, $v = \sin^{-1}\left(\frac{y}{x}\right)$

Sol: Given, $x = e^u \sec v \rightarrow ①$, $y = e^u \tan v \rightarrow ②$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \sec v & e^u \sec v \tan v \\ e^u \tan v & e^u \sec^2 v \end{vmatrix}$$

$$= e^{2u}(\sec^3 v - \sec v \tan^2 v)$$

$$= e^{2u} \sec v (\sec^2 v - \tan^2 v)$$

$$= e^{2u} \sec v\ (1) \quad (\because \sec^2 v - \tan^2 v = 1)$$

$$= e^{2u} \sec v$$

Now, ①^2-②^2 $\Rightarrow$ $x^2-y^2 = e^{2u}\sec^2 v - e^{2u}\tan^2 v$

$$x^2-y^2 = e^{2u}(\sec^2 v - \tan^2 v)$$

$$x^2-y^2 = e^{2u}$$

$$2u = \log(x^2-y^2)$$

$$u = \frac{1}{2}\log(x^2-y^2) \rightarrow ③$$

Now, $\frac{②}{①} \Rightarrow \frac{y}{x} = \frac{e^u \tan v}{e^u \sec v} = \sin v$

$$\frac{y}{x} = \sin v$$

$$v = \sin^{-1}\left(\frac{y}{x}\right) \rightarrow ④$$

For our convinience, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} \frac{x}{x^2-y^2} & \frac{-y}{x^2-y^2} \\ \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$= \frac{x}{(x^2-y^2)^{3/2}} - \frac{y^2}{x(x^2-y^2)^{3/2}}$$

$$= \frac{\cancel{(x^2-y^2)}}{x\cancel{(x^2-y^2)}\sqrt{(x^2-y^2)}}$$

$$= \frac{1}{e^u \sec v\,(e^{2u})^{1/2}}$$

$$= \frac{1}{e^{2u}\sec v}$$

$\therefore$ $J.J' = 1$

Hence proved.

3) for Spherical polar coordinates $x = r\sin\theta.\cos\phi$, $y = r\sin\theta.\sin\phi$

$z = r\cos\theta$. show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

Sol: $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$\frac{\partial x}{\partial r} = \sin\theta.\cos\phi$; $\frac{\partial x}{\partial \theta} = r\cos\theta.\cos\phi$, $\frac{\partial x}{\partial \phi} = -r\sin\theta.\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta.\sin\phi$; $\frac{\partial y}{\partial \theta} = r\cos\theta.\sin\phi$, $\frac{\partial y}{\partial \phi} = r\sin\theta.\cos\phi$

$\frac{\partial z}{\partial r} = \cos\theta$, $\frac{\partial z}{\partial \theta} = -r\sin\theta$, $\frac{\partial z}{\partial \phi} = 0$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta.\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \sin\theta\cos\phi\left(0+r^2\sin^2\theta\cos\phi\right) - r\cos\theta\cos\phi\left(0-r\sin\theta.\cos\theta.\cos\phi\right) - r\sin\theta\sin\phi\left(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi\right)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin^3\theta\cos^2\phi + r^2\cos^2\theta.\cos^2\phi\sin\theta + r^2\sin^3\theta.\sin^2\phi + r^2\cos^2\theta\sin\theta.\sin^2\phi$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta\cos^2\phi\left(\sin^2\theta+\cos^2\theta\right) + r^2\sin\theta\sin^2\phi\left(\sin^2\theta+\cos^2\theta\right)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.\cos^2\phi + r^2\sin\theta.\sin^2\phi$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta\left(\cos^2\phi+\sin^2\phi\right)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

Hence proved.

4) In cylindrical coordinates $x=\rho\cos\phi$, $y=\rho\sin\phi$, $Z=z$,

Show that $\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho$

Sol: Here, $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial z} \end{vmatrix}$

$\dfrac{\partial x}{\partial \rho} = \cos\phi$, $\dfrac{\partial x}{\partial \phi} = -\sin\phi . \rho$, $\dfrac{\partial x}{\partial z} = 0$, $\dfrac{\partial y}{\partial \rho} = \sin\phi$, $\dfrac{\partial y}{\partial \phi} = \rho\cos\phi$

$\dfrac{\partial y}{\partial z} = 0$, $\dfrac{\partial z}{\partial \rho} = 0$, $\dfrac{\partial z}{\partial \phi} = 0$, $\dfrac{\partial z}{\partial z} = 1$.

$$= \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \rho\cos^2\phi + \rho\sin^2\phi$$

$$= \rho\,(\cos^2\phi + \sin^2\phi)$$

$$= \rho(1)$$

$$= \rho .$$

$\therefore \dfrac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho$ // Hence proved.

5) If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ where $x = r\cos\theta$, $y = r\sin\theta$.

Show that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 6r^3\sin 2\theta$

Sol: Given, $u = x^2 - 2y^2$, $v = 2x^2 - y^2$

Here, $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = J\left(\dfrac{u,v}{x,y}\right)$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = -4xy + 16xy = 12xy$$

Also given, $x = r\cos\theta$, $y = r\sin\theta$.

$$J\left(\frac{x,y}{r,\theta}\right) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta + r\sin^2\theta = r(1) = r$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = 12xy \cdot r$$

$$= 12(r\cos\theta)(r\sin\theta)r$$

$$= r^3 6(2\sin\theta\cos\theta)$$

$$\frac{\partial(uv)}{\partial(r,\theta)} = \underline{\underline{6r^3 \sin 2\theta}}$$

Hence proved

6) If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$

<u>Sol</u>: Given, $u = x^2 - y^2$, $v = 2xy$

Here, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4x^2 + 4y^2$$
$$= 4(x^2 + y^2)$$

Also given, $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$
$$= r(1)$$
$$= r$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= 4(x^2+y^2)\cdot r$$

$$= 4r(r^2\cos^2\theta + r^2\sin^2\theta)$$

$$= 4r^2 \cdot r(1)$$

$$= \underline{4r^3} \text{ is required.}$$

7)* If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_1 x_3}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$ then show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

Sol: Given, $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_1 x_3}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$

$$J\left(\frac{y_1, y_2, y_3}{x_1, x_2, x_3}\right) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$J\left(\frac{y_1, y_2, y_3}{x_1, x_2, x_3}\right) = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$J\left(\frac{y_1, y_2, y_3}{x_1, x_2, x_3}\right) = \frac{1}{x_1^2 \cdot x_2^2 \cdot x_3^2} \begin{vmatrix} -x_2 x_3 & +x_1 x_3 & x_2 x_3 \\ x_3 x_2 & -x_1 x_3 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{\cancel{x_1^2 x_2^2 x_3^2}}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= -1(1-1) - 1(-1-1) + 1(1+1)$

$= -1(0) - 1(-2) + 1(2)$

$= 2+2$

$= \underline{\underline{4}}$

Hence proved

8) If $x = u(1-v)$, $y = uv$ then prove that $JJ' = 1$

Sol: Given, $x = u(1-v) \rightarrow ①$, $y = uv \rightarrow ②$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$J = u - \cancel{uv} + \cancel{uv}$$

$$J = u$$

From ① $x = u - uv$

$x = u - y$ ($\because$ from ②)

$\boxed{u = x+y}$

From ② $y = uv$

$v = \frac{y}{u}$

$\boxed{v = \frac{y}{x+y}}$

$$\text{Here, } J'\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$v = (y)\left(\frac{1}{x+y}\right)$ ($\because \frac{d}{dx} uv$)

$$\frac{\partial v}{\partial y} = y\left(\frac{-1}{(x+y)^2}\right) + \frac{1}{(x+y)}(1)$$

$$\frac{\partial v}{\partial y} = \frac{-\cancel{y} + x + \cancel{y}}{(x+y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{x}{(x+y)^2}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{\cancel{(x+y)}}{(x+y)^{\cancel{2}}}$$

$$= \frac{1}{x+y}$$

$$= \frac{1}{u}$$

$\therefore JJ' = u\left(\frac{1}{u}\right) = \underline{1}$ Hence proved

1) If $u = x+y+z$, $uv = y+z$, $uvw = z$ show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

Sol: Given, $u = x+y+z \to (1)$, $uv = y+z \to (2)$, $uvw = z \to (3)$

$(1) \Rightarrow u = x+y+z$

$u = x + uv$

$x = u - uv$

$x = u(1-v)$

$(2) \Rightarrow uv = y+z$

$uv = y + uvw$

$y = uv - uvw$

$y = uv(1-w)$

$(3) \Rightarrow z = uvw$

$$\therefore J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$J\left(\frac{x,y,z}{u,v,w}\right) = (1-v)\left[u^2v - \cancel{u^2wv} + \cancel{u^2vw}\right] + u\left[uv^2 - \cancel{v^2uw} + \cancel{uv^2w}\right]$$

$$J\left(\frac{x,y,z}{u,v,w}\right) = u^2v - \cancel{u^2v^2} + \cancel{u^2v^2} - \cancel{v^2uw} + \cancel{uv^2w}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$$

Hence proved.

⇒ FUNCTIONAL DEPENDENCE:

→ Two functions 'u' & 'v' are functionally dependent if their Jacobian i.e., $J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$.

→ If $J\left(\frac{u,v}{x,y}\right) \neq 0$ then 'u', 'v' are said to be functionally independent.

Similarly, $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$ then u, v, w are functionally dependent

Otherwise functionally independent.

20/1/2020

1) If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$. Find $\frac{\partial(u,v)}{\partial(x,y)}$ are u and v functionally related? If so find the relationship.

Sol: Given, $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Now,

$$\frac{\partial u}{\partial x} = (x+y) \cdot \frac{-1}{(1-xy)^2}(-y) + 1 \cdot \frac{1}{(1-xy)}$$

$$\Rightarrow \frac{xy + y^2 + (1-xy)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = (x+y) \cdot \frac{-1}{(1-xy)^2}(-x) + 1 \cdot \frac{1}{(1-xy)}$$

$$\Rightarrow \frac{(x^2+xy) + (1-xy)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{(1+y^2)}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{(1+y^2)}{(1-xy)^2(1+y^2)} - \frac{(1+x^2)}{(1+x^2)(1-xy)^2}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= 0$$

$\therefore$ u and v are functionally related.

Here $V = \tan^{-1}x + \tan^{-1}y$

$V = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$V = \tan^{-1}(u)$

$\boxed{u = \tan V}$

2) Show that $u = y+z$, $v = x+2z^2$, $w = x-4yz-2y^2$ are not independent. Find the relationship between them.

Sol: $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4z-4y & -4y \end{vmatrix}$

$= 0(4z(4z-4y)) - 1(-4y-4z) + 1(-4z-4y-0)$

$= 4y + 4z - 4z - 4y$

$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

∴ u, v, w are functionally related.

Now, $u^2 = (y+z)^2 = y^2 + z^2 + 2yz$

$w = x - 4yz + 2y^2$

$w = x - 2(2yz + y^2)$

$w = x + 2z^2 - 2(2yz + z^2 + y^2)$

$w = (x+2z)^2 - 2(z+y)^2$

$w = V - 2u^2$

∴ $\boxed{w = V - 2u^2}$

3) If

that

Sol: G

3) If $u = 3x+2y-z$, $v = x-2y+z$ & $w = x(x+2y-z)$ then show that they are functionally related and find the relation.

Sol: Given, $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 2x+2y-z & 2x & -x \end{vmatrix}$

$= 3(2x-2x) - 2(-x-2x-2y+z) - 1(2x+2(2x+2y-z))$

$= \cancel{6x} + \cancel{4y} - \cancel{2z} - \cancel{6x} - \cancel{4y} + \cancel{2z}$

$= 0$

$\therefore$ u, v, w are functionally related

$u^2 - v^2 = (3x+2y-z)^2 - (x-2y+z)^2$

$u^2 - v^2 = 9x^2 + 4y^2 + z^2 + 12xy - 4yz - 6xz - (x^2 + 4y^2 + z^2 - 4xy - 4yz + 2xz)$

$u^2 - v^2 = 8x^2 + 16xy - 8xz$

$u^2 - v^2 = 8(x^2 + 2xy - xz)$

$u^2 - v^2 = 8x(x+2y-z)$

$u^2 - v^2 = 8w$

4) Determine whether the ~~functi~~ following functions are functionally dependent or not. If so, find the relation.

$u = \sin^{-1}x + \sin^{-1}y$, $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$.

Sol: Given, $u = \sin^{-1}x + \sin^{-1}y$, $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} + y \cdot \frac{1(-2x)}{2\sqrt{1-x^2}} & \frac{x(-2y)}{2\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= \cancel{1} - \cancel{\frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}} - \cancel{1} + \cancel{\frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}}$$

$$= 0$$

$\therefore$ u and v are functionally related

$$u = \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

$$u = \sin^{-1}(v)$$

$$\therefore u = \sin^{-1}(v)$$

$$\boxed{v = \sin u}$$

5) Verify if $u = 2x - y + 3z$, $v = 2x - y - z$, $w = 2x - y + z$ are functionally dependent. And if so, find the relationship between them. $(u+v) = 2w$

6) Show that the following functions are functionally dependent. Hence find the functional relationship between them. $u = x e^y \sin z$, $v = x \cdot e^y \cdot \cos z$, $w = x^2 \cdot e^{2y}$. $u^2+v^2 = w$

5) Given, $u = 2x - y + 3z$, $v = 2x - y - z$, $w = 2x - y + z$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= 2(-1-1) + 1(2+2) + 3(\cancel{-2}+\cancel{2})$$

$$= -4 + 4$$

$$= 0.$$

$\therefore$ u and v are functionally related.

$u+v = 2x-y+3z+2x-y+z$

$u+v = 4x-2y+2z$

$u+v = 2(2x-y+z)$

$\therefore \boxed{u+v = 2w}$

6) Given, $u = x.e^y \sin z$, $v = x.e^y \cos z$, $w = x^2.e^{2y}$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} e^y \sin z & xe^y \sin z & xe^y \cos z \\ e^y \cos z & xe^y \cos z & -xe^y \sin z \\ 2xe^{2y} & 2x^2e^{2y} & 0 \end{vmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 2xe^{2y}(-x^2e^{2y}\sin^2 z - x^2e^{2y}\cos^2 z) - 2x^2e^{2y}(-xe^{2y}\sin^2 z - xe^{2y}\cos^2 z)$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2x^3e^{4y}(\sin^2 z + \cos^2 z) + 2x^3e^{4y}(\sin^2 z + \cos^2 z)$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2x^3e^{4y} + 2x^3e^{4y}$$

$$= 0$$

$\therefore$ u and v are functionally related.

$u^2+v^2 = x^2e^{2y}\sin^2 z + x^2e^{2y}\cos^2 z$

$u^2+v^2 = x^2e^{2y}(\sin^2 z + \cos^2 z)$

$u^2+v^2 = x^2e^{2y}$

$\therefore \boxed{u^2+v^2 = w}$

If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = xy+yz+zx$, show that they are functionally related and find the relation b/w them.

22-01-2020

MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES.

Let $f(x,y)$ be a function of two variables (x,y), then function $f(x,y)$ is said to have maximum value or minimum value at (a,b) if

if $f(a,b) > f(a+h, b+k)$ (or) where $h,k > 0$

$f(a,b) < f(a+h, b+k)$ for small values of (h,k)

Stationary point: A point (a,b) is said to be stationary point of $f(x,y)$ if $\frac{\partial f(a,b)}{\partial x} = 0, \frac{\partial f(a,b)}{\partial y} = 0$.

Extreme point: The stationary point of $f(x,y)$ is said to be an extreme point if it is either point of minima or point of maxima.

Saddle point: A stationary point is said to be if it is neither point of minima nor point of maxima.

→ At Saddle point, $f(x,y)$ is minimum in one direction while maximum in another direction.

→ The neccessary conditions for $f(x,y)$ to have a maximum or minimum are $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$.

Working rule to find maxima or minima of $f(x,y)$:

1) Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and equate them to '0' to get stationary points.

Let the stationary points be (a_1, b_1); (a_2, b_2); (a_3, b_3); ....

2) Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

(i) if $rt - s^2 > 0$ & $r < 0$ at (a_1, b_1) then 'f' attains maximum value at (a_1, b_1) and maximum value of $f = f_{max} = f(a_1, b_1)$

(ii) if $rt - s^2 > 0$ & $r > 0$ at (a_2, b_2) then 'f' attains minimum value at (a_2, b_2) and minimum value of $f = f_{min} = f(a_2, b_2)$

(iii) if $rt - s^2 < 0$ at (a_3, b_3) then 'f' attains neither maxima nor minima at (a_3, b_3) and also (a_3, b_3) called saddle point

(iv) if $rt - s^2 = 0$ at (a_4, b_4) then no conclusion can be drawn about maxima or minima of $f(x,y)$ at (a_4, b_4). It needs further investigation.

I) Find the minimum and maximum values of

1) $x^3 + y^3 - 3axy$ 2) $2(x^2 - y^2) - x^4 y^4$ 3) $f(x,y) = x^3 y^2 (1 - x - y)$

4) $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ 5) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

1) Let $f(x,y) = x^3 + y^3 - 3axy$

For maxima or minima, we have $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.

i.e., $\frac{\partial f}{\partial x} = 3x^2 - 3ay = 0$, $\frac{\partial f}{\partial y} = 3y^2 - 3ax = 0$

$$x^2 - ay = 0 \rightarrow (1) \quad , \quad y^2 - ax = 0 \rightarrow (2)$$

from (1) $y = \frac{x^2}{a} \rightarrow (3)$

Sub (3) in (2), we get $\frac{x^4}{a^2} - ax = 0$

$$x^4 - a^3 x = 0$$

$$x(x^3 - a^3) = 0$$

$$x = 0, x = a$$

from (3), when $x = 0$, $y = 0$

when $x = a$, $y = a$

$\therefore (0,0), (a,a)$ are stationary points of $f(x,y)$

Now, $r = \frac{\partial^2 f}{\partial x^2} = 6x$, $s = \frac{\partial^2 f}{\partial x \partial y} = -3a$, $t = \frac{\partial^2 f}{\partial y^2} = 6y$

$rt - s^2 = 36xy - 9a^2$, $r = 6x$.

At $(0,0)$, $rt - s^2 = 0 - 9a^2 = -9a^2 < 0$, then 'f' attains neither maxima nor minima at $(0,0)$.

$\therefore (0,0)$ is a saddle point.

At (a,a), $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

$r = 6a > 0$

then 'f' attains minimum at (a,a)

$\therefore$ Min. value $= f_{min} = f(a,a) = a^3 + a^3 - 3a^3 = -a^3$ //

(2) Let $f(x,y) = 2(x^2 - y^2) - x^4 + y^4$.

For maxima or minima we have $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 4x - 4x^3 = 0$ $\qquad$ $\frac{\partial f}{\partial y} = -4y + 4y^3 = 0$

$x - x^3 = 0$ $\qquad$ $y^3 - y = 0$

$x(1 - x^2) = 0$ $\qquad$ $y(y^2 - 1) = 0$

$x = 0$, $x = \pm 1$ $\qquad$ $y = 0$, $y = \pm 1$

$\therefore (0,0)\ (0,1)\ (0,-1)\ (1,0)\ (1,1)\ (1,-1)\ (-1,0)\ (-1,1)\ (-1,-1)$

are stationary points of $f(x,y)$

$r = \frac{\partial^2 f}{\partial x^2} = 4 - 12x^2$, $s = \frac{\partial^2 f}{\partial x \partial y} = 0$, $t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

$rt - s^2 = 48y^2 - 16 - 144x^2y^2 + 48x^2$

$rt - s^2 = 48x^2 + 48y^2 - 144x^2y^2 - 16$, $r = 4 - 12x^2$

$\rightarrow$ at $(0,0)$, $rt - s^2 = -16 < 0$

Then 'f' attains neither min. nor max. at $(0,0)$

$\therefore$ (1,0) is saddle point.

$\rightarrow$ at (0,1), $rt-s^2 = (4-0)(12-4) = 32 > 0$, $r = 4 > 0$, then 'f' attains min value at (0,1)

The min. value $= f_{min} = f(0,1) = 2(0-1) - 0 + 1 = -1$

$\rightarrow$ at (0,-1), $rt-s^2 = 32 > 0$, $r = 4 > 0$ then 'f' attains min. value at (0,-1)

The min. value $= f_{min} = f(0,-1) = 2(0-1) - 0 + 1 = -1$

$\rightarrow$ at (1,0), $rt-s^2 = 32 > 0$, $r = -8 < 0$, then 'f' attains max. value at (1,0).

The max. value $= f_{max} = f(1,0) = 2(1-0) - 1 + 0 = 1$

$\rightarrow$ at (1,1), $rt-s^2 = -64 < 0$, $r = 4 > 0$ then 'f' attains neither min nor max at (1,1).

(1,1) is a saddle point.

$\rightarrow$ (1,-1), $rt-s^2 = -64 < 0$, then 'f' attains neither min nor max at (1,-1).

(1,-1) is a saddle point.

$\rightarrow$ (-1,-1), $rt-s^2 = -64 < 0$, then 'f' attains neither min. nor max. at (-1,-1).

(-1,-1) is a saddle point.

$\rightarrow$ (-1,0), $rt-s^2 = 32 > 0$, and $r = -8$, then 'f' attains max. value at (-1,0).

The max. value $= f(-1,0) = 2(1-0) - 1 + 0 = 1$

$\rightarrow$ (-1,1), $rt-s^2 = -64 < 0$, then 'f' attains neither min nor max at (-1,1).

(-1,1) is a saddle point.

3) Let $f(x,y) = x^3y^2(1-x-y)$

$f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

For maxima or minima we have $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0, \quad \frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2 = 0$$

4) Let $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima or minima, we have $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 4x^3 - 4\overset{x}{\cancel{y}} + 4y = 0 \quad ; \quad \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

$$x^3 - x + y = 0 \rightarrow (1), \qquad y^3 + x - y = 0 \rightarrow (2)$$

Solving (1), (2),

$$\begin{array}{r} x^3 - x + \cancel{y} = 0 \\ y^3 + \cancel{x} - y = 0 \\ \hline x^3 + y^3 = 0 \end{array}$$

$\Rightarrow x^3 + y^3 = (x+y)(x^2 - xy + y^2) = 0$

$\Rightarrow x = -y \rightarrow (3)$ (or) $x^2 - xy + y^2 = 0.$

from (1), $x^3 - x - x = 0 \quad (\because y = -x)$

$x^3 - 2x = 0$

$x(x^2 - 2) = 0$

$x = 0$ (or) $x = \pm\sqrt{2}$

From (3), when $x = 0$, $y = 0$ $\quad (0,0)$

$x = \sqrt{2}$, $y = -\sqrt{2}$ $\quad (\sqrt{2}, -\sqrt{2})$

$x = -\sqrt{2}$, $y = \sqrt{2}$ $\quad (-\sqrt{2}, \sqrt{2})$

$\therefore (0,0)\ (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$ are stationary points of $f(x,y)$

$r = 12x^2 - 4$, $s = 4$, $t = 12y^2 - 4$

$\Rightarrow$ ~~at~~ $rt - s^2 = (12x^2 - 4)(12y^2 - 4) - 16$

$r = 12x^2 - 4$

$\rightarrow$ at $(0,0)$, $rt - s^2 = 0$, then no conclusion drawn about min or max at $(0,0)$

$=0$

$\rightarrow$ at $(\sqrt{2}, -\sqrt{2})$, $rt-s^2 = 384 > 0$, $r = 20 > 0$ then 'f' attains minimum value at $(\sqrt{2}, -\sqrt{2})$

The min. value $= f(\sqrt{2}, -\sqrt{2}) = (\sqrt{2})^4 + (\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(2) - 2(2)$

$$= 4 + 4 - \cancel{4} + 8 - \cancel{4}$$

$$= -8$$

$\rightarrow$ at $(-\sqrt{2}, \sqrt{2})$, $rt-s^2 = 384$ and $r = 20$, then 'f' attains minimum value at $(-\sqrt{2}, \sqrt{2})$

$\therefore$ The min. value $= f_{min} = f(-\sqrt{2}, \sqrt{2}) = -8$.

3) Let $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$.

for minimor maxima we have $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$; $\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2 = 0$

$x^2y^2(3-4x-3y) = 0 \rightarrow (1)$; $x^3y(2-2x-3y) = 0 \rightarrow (2)$

(1) $\Rightarrow$ $x=0$, $y=0$, $3-4x-3y=0$

(2) $\Rightarrow$ $x=0$ or $y=0$, $2-2x-3y=0$.

$$\begin{array}{r} 3-4x-\cancel{3y}=0 \\ 2-2x-\cancel{3y}=0 \\ \hline x=\frac{1}{2}, y=\frac{1}{3} \end{array}$$

$(0,0)$, $\left(\frac{1}{2}, \frac{1}{3}\right)$ are stationary points.

Now, $r = 6xy^2 - 12x^2y^2 - 6xy^3$, $s = 6x^2y - 8x^3y - 9x^2y^2$.

$t = 2x^3 - 2x^4 - 6x^3y$.

$rt - s^2 = (6xy^2 - 12x^2y^2 - 6xy^3)(\cancel{6x^2y - 8x^3}\ 2x^3 - 2x^4 - 6x^3y)$

$- (6x^2y - 8x^3y - 9x^2y^2)$

$\rightarrow$ at $(0,0)$, $rt - s^2 = 0$, then no conclusion drawn about min or max at $(0,0)$.

$\rightarrow$ at $\left(\frac{1}{2}, \frac{1}{3}\right)$, $rt-s^2 = \left(\frac{1}{3} - \frac{1}{3} - \frac{1}{9}\right)\left(\frac{1}{4} - \frac{1}{8} - \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)^2$

$$= \left(\frac{-1}{9}\right)\left(\frac{-1}{8}\right) - \left(\frac{1}{12}\right)^2$$

$$= \frac{1}{72} - \frac{1}{144}$$

$$= \frac{1}{144} > 0 \qquad r = \frac{-1}{9} < 0$$

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then 'f' attain max value at $(\frac{1}{2},\frac{1}{2})$ and $f_{max} = f(\frac{1}{2},\frac{1}{2})$

$$= \frac{1}{472}\left(\frac{1}{6}\right)$$

$$= \frac{1}{432}$$

5) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

For maxima or minima, we have $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 = 0$$

$$x^2 + y^2 - 10x + 24 = 0$$

$y = 0$ $\quad x^2 - 10x + 24 = 0$

$(6,0)\ (4,0)$

$$\frac{\partial f}{\partial y} = 6xy - 30y = 0$$

$$xy - 5y = 0$$

$$y(x-5) = 0$$

$$y = 0,\ x - 5 = 0$$

$$x = 5$$

for $x = 5$

$$25 + y^2 - 50 + 24 = 0$$

$$y^2 - 1 = 0$$

$$y = \pm 1$$

$(5,1)\ (5,-1)$

$(6,0)\ (4,0)\ (5,1)\ (5,-1)$ are stationary points.

$r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$, $\quad s = \frac{\partial^2 f}{\partial x \partial y} = 6y$

$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$.

$rt - s^2 = (6x-30)^2 - 36y^2$, $r = 6x - 30$

→ at $(6,0)$, $rt - s^2 = (36-30) = 36 > 0$, $r = 36 - 30 = 6 > 0$

'f' attains minimum value and $f_{min} = f(6,0)$

$$f_{min} = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

$$f_{min} = 108$$

→ at $(4,0)$, $rt - s^2 = (24-30)^2 = 36 > 0$, $\quad r = 6(4) - 30$, $r = -6 < 0$

'f' attains maximum value and $f_{max} = f(4,0)$

$$f_{max} = 64 + 0 - 15(16) + 72(4)$$

$$f_{max} = 112$$

at (5,1)
'f' atta
and (5,
at (5,-
'f' atta
6) In c
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at $(5,1)$, $rt-s^2 = 0-36 = -36<0$

'f' attains neither min nor max at $(5,1)$

and $(5,1)$ is called saddle point.

at $(5,-1)$, $rt-s^2 = -36<0$

'f' attains neither min or max at $(5,-1)$

6)* In a triangle find the maximum value of $\cos A \cos B \cos C$.

Sol: In a triangle, we have $A+B+C = \pi$

$$C = \pi - (A+B)$$

$$\cos A \cos B \cos C = \cos A \cos B \cos(\pi-(A+B))$$

$$= -\cos A \cos B \cos(A+B)$$

$$= f(A,B).$$

$\therefore$ $f(A,B) = -\cos A \cos B . \cos(A+B)$

For minima or maxima we have $\frac{\partial f}{\partial A} = 0$, $\frac{\partial f}{\partial B} = 0$.

i.e. $\frac{\partial f}{\partial A} = -\cos B\left(\cos A(-\sin(A+B)) - \sin A . \cos(A+B)\right) = 0$

$\frac{\partial f}{\partial A} = +\cos B\left(\sin A . \cos(A+B) + \cos A . \sin(A+B)\right) = 0$.

$\frac{\partial f}{\partial A} = \cos B\left(\sin(A+(A+B))\right) = 0$ $(\sin\theta = \sin n\pi)$

$\cos B = 0$, $\sin(2A+B) = 0$ for triangle $n=1$

$B = \frac{\pi}{2}$, $\sin(2A+B) = \sin\pi$

$2A+B = \pi \rightarrow (1)$

$\frac{\partial f}{\partial B} = \cos A\left(\sin(A+2B)\right) = 0$

$\cos A = 0$, $\sin(A+2B) = 0$

$A = \frac{\pi}{2}$, $A+2B = \pi \rightarrow (2)$

Solving (1) and (2), we get $A = \frac{\pi}{3}$, $B = \frac{\pi}{3}$.

$\therefore$ $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is stationary point of $f(A,B)$.

Now, $r = 2\cos B . \cos(2A+B)$, $s = \cos B . \cos(2A+B) - \sin B . \sin(2A+B)$

$s = \cos(B+2A+B)$

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$s = \cos(2A+2B)$, $t = 2\cos A . \cos(A+2B)$

$rt - s^2 = 4\left(\cos B . \cos(2A+B)\right)\left(\cos A . \cos(A+2B)\right) - \left(\cos^2(2A+2B)\right)$

$r = 2\cos B . \cos(2A+B)$

$\rightarrow$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $rt - s^2 = 4\left(\frac{1}{2}(-1)\right)\left(\frac{1}{2}\times(-1)\right) - \left(\frac{1}{2}\right)^2$

$= 1 - \frac{1}{4}$

$= \frac{3}{4} > 0$

$r = 2\left(\frac{1}{2}\right)\times(-1) = -1 < 0$

then 'f' attains maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

The max. value $= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{\pi}{3}$

$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$

$= \frac{1}{8}$

we have $C = \pi - (A+B)$

$C = \pi - \frac{2\pi}{3}$

$\boxed{A = B = C = \frac{\pi}{3}}$

$\therefore$ Given triangle is equilateral triangle

$\Rightarrow$ Lagrange's Method of Undetermined Multipliers:

This method is useful to find the extreme values (minima or maxima) of the function of three variables, subject to the condition given.

Procedure:

Let $f(x,y,z)$ be the given function of three variables x, y, z subject to the condition $\phi(x,y,z) = 0$. $\rightarrow$ ①

(which are connected by the relation). Equation

1) consider the lagrangian function (auxiliary function)

$$F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$$

where 'λ' is a lagrangian multiplier.

2) Find the values of $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$.

3) Using $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ and equ ① find the value of 'λ'. Hence calculate the values of x, y, z.

1) Find the minimum value of $x^2+y^2+z^2$ subject to the condition (i) $ax+by+cz = p$ (ii) $xyz = a^3$.

Sol: Let $f(x,y,z) = x^2+y^2+z^2$ and

(i) $\phi(x,y,z) = ax+by+cz-p = 0 \rightarrow ①$

Consider the lagrangian function (auxiliary equation)

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$$

$$F(x,y,z) = x^2+y^2+z^2+\lambda(ax+by+cz-p)$$

For minima or maxima, we have.

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0.$$

$$2x+a\lambda = 0, \quad 2y+b\lambda = 0, \quad 2z+c\lambda = 0.$$

$$x = -\frac{a\lambda}{2} \rightarrow ②, \quad y = -\frac{b\lambda}{2} \rightarrow ③, \quad z = -\frac{c\lambda}{2} \rightarrow ④.$$

Sub ②, ③ & ④ in ①, we get.

$$a\left(-\frac{a\lambda}{2}\right) + b\left(-\frac{b\lambda}{2}\right) + c\left(-\frac{c\lambda}{2}\right) - p = 0$$

$$-\frac{\lambda}{2}(a^2+b^2+c^2) - p = 0$$

$$-a^2\lambda - b^2\lambda - c^2\lambda - 2p = 0$$

$$-\lambda(a^2+b^2+c^2) = 2p$$

$$\lambda = \frac{-2p}{a^2+b^2+c^2} \rightarrow ⑤$$

Sub ⑤ in ②, ③ & ④ we get.

$$x = -\frac{a}{2} \times \frac{-2p}{a^2+b^2+c^2} = \frac{ap}{a^2+b^2+c^2}$$

Similarly, $y = \dfrac{bp}{a^2+b^2+c^2}$, $z = \dfrac{cp}{a^2+b^2+c^2}$

$\therefore f(x,y,z) = x^2+y^2+z^2$ has min. value at

$$\therefore f\left(\frac{ap}{a^2+b^2+c^2}, \frac{bp}{a^2+b^2+c^2}, \frac{cp}{a^2+b^2+c^2}\right) \text{ is}$$

$$f = \frac{a^2p^2}{(a^2+b^2+c^2)^2} + \frac{b^2p^2}{(a^2+b^2+c^2)^2} + \frac{c^2p^2}{(a^2+b^2+c^2)^2}$$

$$f = \frac{\cancel{(a^2+b^2+c^2)}\,p^2}{(a^2+b^2+c^2)^{\cancel{2}}}$$

$$f = \frac{p^2}{(a^2+b^2+c^2)}$$

(ii) Let $f(x,y,z) = x^2+y^2+z^2$ and

$\phi(x,y,z) = xyz - a^3 = 0 \rightarrow (1)$

Consider the lagrangian function (auxiliary equation)

$F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$

$F(x,y,z) = x^2+y^2+z^2 + \lambda(xyz - a^3)$

For minima or maxima, we have $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0$

$2x + \lambda yz = 0$, $2y + \lambda xz = 0$, $2z + \lambda xy = 0$

$2x = -\lambda yz$ $\qquad$ $2y = -\lambda xz$ $\qquad$ $2z = -\lambda xy$

$\dfrac{x}{yz} = \dfrac{-\lambda}{2} \rightarrow (2)$ $\qquad$ $\dfrac{y}{xz} = \dfrac{-\lambda}{2} \rightarrow (3)$ $\qquad$ $\dfrac{z}{xy} = \dfrac{-\lambda}{2} \rightarrow (4)$

From (2), (3) $\quad \dfrac{x}{y\cancel{z}} = \dfrac{y}{x\cancel{z}}$

$x^2 = y^2 \rightarrow (5)$

from (3), (4

from (5), (6

Sub (7)

$\therefore$ x

$\therefore$ f

f

2) Find

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from ③, ④ $\frac{y}{xz} = \frac{z}{xy}$

$y^2 = z^2 \rightarrow ⑥$

from ⑤, ⑥ $x^2 = y^2 = z^2 \Rightarrow x = y = z \rightarrow ⑦$

Sub ⑦ in ① $x^3 - a^3 = 0$

$x^3 = a^3$

$x = a$

$\therefore \quad x = y = z = a$

$\therefore \; f(x,y,z) = x^2+y^2+z^2$ has min. value at $f(a,a,a)$ is

$f = a^2 + a^2 + a^2$

$f = 3a^2$

2) Find the points on the surface $z^2 = xy + 1$ nearest to the origin also find the shortest distance from origin to the surface.

sol: Let $P(x,y,z)$ be the point on the surface which is near to the origin then the distance from the origin to the point on the surface is $d = \sqrt{x^2+y^2+z^2}$

Here $f(x,y,z) \equiv d^2 = x^2+y^2+z^2$

$\phi(x,y,z) = z^2 - xy - 1 = 0 \rightarrow ①$

consider the lagrangian function (auxiliary equation)

$F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$.

$F(x,y,z) = x^2+y^2+z^2 + \lambda\,(z^2 - xy - 1)$

For minima or maxima, we have $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

i.e., $2x - \lambda y = 0 \rightarrow ②$, $2y - \lambda x = 0 \rightarrow ③$, $2z + 2\lambda z = 0 \rightarrow ④$

$\boxed{\lambda = -1}$

Sub ④ in ② and ③, we get

$2x + y = 0 \rightarrow ⑤$, $2y + x = 0 \rightarrow ⑥$

Solving ⑤, ⑥, we get $x = 0, y = 0$.

Sub these values in equ ①, we get

$$z^2 - 0 - 1 = 0$$
$$z^2 = 1$$
$$z = \pm 1$$

$\therefore (x, y, z) = (0, 0, \pm 1)$

$\therefore$ The points on the surface are $(0, 0, 1)$ and $(0, 0, -1)$ and the shortest distance from origin to the point on the surface is $d = \sqrt{0 + 0 + 1}$

$\boxed{d = 1}$ unit

3) Given $x + y + z = a$, find the maximum of $x^m y^n z^p$.

Sol: Let $f(x, y, z) = x^m y^n z^p$ and $\phi(x, y, z) = x + y + z - a = 0$ →①

The Lagrangian function (Auxiliary equation)

$$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$$

$$F(x, y, z) = x^m y^n z^p + \lambda (x + y + z - a)$$

For maximum or minimum, we have $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

i.e., $m x^{m-1} y^n z^p + \lambda = 0$, $n x^m y^{n-1} z^p + \lambda = 0$, $p x^m y^n z^{p-1} + \lambda = 0$

$$\frac{m}{x} x^m y^n z^p + \lambda = 0, \quad \frac{n}{y} x^m y^n z^p + \lambda = 0, \quad \frac{p}{z} x^m y^n z^p + \lambda = 0$$

$$\frac{m}{x} \cdot f + \lambda = 0, \quad \frac{n}{y} \cdot f + \lambda = 0, \quad \frac{p}{z} f + \lambda = 0 \quad (\because f = x^m y^n z^p)$$

$\lambda = -\frac{m}{x} f \Rightarrow x = -\frac{m}{\lambda} f$ →② Similarly $y = -\frac{n}{\lambda} f$ →③, $z = -\frac{p}{\lambda} \cdot f$ →④

Sub ②, ③, ④ in ①

$$-\frac{m}{\lambda} f - \frac{n}{\lambda} f - \frac{p}{\lambda} f - a = 0$$

$$-\frac{f}{\lambda} (m + n + p) = a$$

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$\lambda = \frac{-f}{a}(m+n+p) \rightarrow (5)$

Sub (5) in (2), (3), (4)

$(2) \Rightarrow x = +\frac{ma \cdot f}{f(m+n+p)} = \frac{ma}{m+n+p}$

Similarly $(3) \Rightarrow \frac{na}{m+n+p} = y$, $(4) \Rightarrow z = \frac{pa}{m+n+p}$

$\therefore f(x,y,z) = x^m y^n z^p$ has max. value at $f\left(\frac{ma}{m+n+p}, \frac{na}{m+n+p}, \frac{pa}{m+n+p}\right)$

is $f = \frac{m^m a^m}{(m+n+p)^m} \cdot \frac{n^n a^n}{(m+n+p)^n} \cdot \frac{p^p a^p}{(m+n+p)^p}$

$$f = \frac{m^m n^n p^p a^{m+n+p}}{(m+n+p)^{m+n+p}}$$

(4) A rectangular box opened at the top is to have volume of 32 c.ft (cubic feet). Find the dimensions of the box requiring least material for its construction.

Sol: Let x ft, y ft, z ft be the dimensions of the rectangular box and 's' be the surface of the box, then

$S(x,y,z) = xy + 2yz + 2zx$ ($\because$ Box is opened at one top).

Also given that, volume of the box is 32 c.ft.

i.e., $xyz = 32$

Let $\phi(x,y,z) = xyz - 32 = 0 \rightarrow (1)$

The Lagrangian function

$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$

i.e., $F(x,y,z) = \lambda(xyz - 32) + xy + 2yz + 2zx$.

For max or min. we have, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$.

$y + 2z + \lambda yz = 0 \rightarrow (2)$ $\quad x + 2z + \lambda xz = 0 \rightarrow (3)$ $\quad 2y + 2x + \lambda xy = 0 \rightarrow (4)$

Multiplying equ (2) by x, (3) by y, (4) by z, we get

$xy + 2xz + \lambda xyz = 0 \rightarrow (5)$

$xy + 2yz + \lambda xyz = 0 \rightarrow (6)$

$2yz + 2xz + \lambda xyz = 0 \rightarrow (7)$

(5) - (6) gives $2xz - 2yz = 0$; (6) - (7) gives $xy - 2xz = 0$

$2z(x-y) = 0$; $x(y-2z) = 0$

$z \neq 0,\ x-y=0$; $x \neq 0,\ y = 2z$

$x = y$; $z = \frac{y}{2} \rightarrow (9)$

$\rightarrow (8)$

Sub. (8) and (9) in equ (1), we get $y.y.\frac{y}{2} - 32 = 0$

$y^3 - 64 = 0$

$\boxed{y = 4}$

from (8), $x = 4$

from (9), $z = \frac{4}{2} = 2$

$\therefore x = 4, y = 4, z = 2$

Hence the dimensions of the box are 4ft, 4ft & 2ft.

5) The sum of 3 numbers is constant. Prove that their product is maximum when they are equal.

Sol: Let x, y, z be 3 numbers such that $x + y + z = a$

also product of 3 numbers is xyz.

let $f(x,y,z) = xyz$ and $\phi(x,y,z) = x+y+z-a = 0 \rightarrow (1)$

The lagrangian function (Auxiliary equation)

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$F(x,y,z) = xyz + \lambda(x+y+z-a)$

For maximum or minimum, we have $\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0$

i.e., $yz + \lambda = 0,\quad xz + \lambda = 0,\quad xy + \lambda = 0$

$yz = -\lambda \rightarrow (2)\quad xz = -\lambda \rightarrow (3)\quad xy = -\lambda \rightarrow (4)$

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from ② and ③ $yz = xz$

$\boxed{x = y} \rightarrow ⑤$

from ③ & ④ $xz = xy$

$\boxed{y = z} \rightarrow ⑥$

from ⑤ and ⑥ $x = y = z \rightarrow ⑦$

Sub ⑦ in ⑪

$\phi(x,y,z) = x + x + x - a = 0$

$3x = a$

$x = \frac{a}{3}$

$\therefore \boxed{x = y = z = \frac{a}{3}}$

$\therefore F(x,y,z)$ has max at $f\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$

Thus the product of 3 numbers is maximum when they are equal. i.e., $\left(\frac{a}{3}\right)\left(\frac{a}{3}\right)\left(\frac{a}{3}\right) = \frac{a^3}{27}$

6) Divide 24 into 3 parts such that the continued product of the first, square of the second and the third. is maximum (Hint: $x+y+z = 24$, $f = xy^2z^3$)

7) Find the dimensions of the rectangular box opened at the top of maximum capacity whose surface is 432 sq.m ($xy + 2yz + 2zx - 432 = 0$, $f(x,y,z) = xyz$)

6) sol: Let x, y, z be the 3 parts such that $x+y+z = 24$

also product of first, square of second, cube of third is xy^2z^3.

Let $f(x,y,z) = xy^2z^3$ and $\phi(x,y,z) = x+y+z-24 = 0 \rightarrow ①$

The lagrangian function (Auxiliary equation)

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$F(x,y,z) = xy^2z^3 + \lambda(x+y+z-24)$

For maximum or minimum, we have $\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0$

i.e., $y^2z^3+\lambda=0, 2yxz^3+\lambda=0, 3xy^2z^2+\lambda=0$

$y^2z^3=-\lambda \rightarrow (2)$, $2yxz^3=-\lambda \rightarrow (3)$, $3xy^2z^2+\lambda=0$

$3xy^2z^2=-\lambda \rightarrow (4)$

From (2) & (3) $\cancel{y^2}\cancel{z^3} = 2\cancel{y}x\cancel{z^3}$

$y = 2x \Rightarrow x = \frac{y}{2} \rightarrow (5)$

From (3) and (4) $2\cancel{y}\cancel{x}z^3 = 3\cancel{x}y^{\cancel{2}}\cancel{z^2}$

$2z = 3y$

$z = \frac{3y}{2} \rightarrow (6)$

Sub (5), (6) in (1)

$\phi(x,y,z) = x+y+z-24 = 0$

$\Rightarrow \frac{y}{2}+y+\frac{3y}{2}-24=0$

$y+2y+3y-48=0$

$6y=48$

$\boxed{y=8}$

$\therefore x=\frac{8}{2}=4, \quad z=\frac{3(8)}{2}=12$

$\therefore x=4, y=8, z=12.$

$\therefore$ The continued product is maximum.

Hence proved.

The max value of $f(x,y,z)$ is $f(4,8,12)$

i.e., $f(4,8,12) = 48^2 12^3$

$= 442368$

7) let x, y, z be the dimensions of the rectangular box

let $f(x,y,z) = xyz$, $\phi(x,y,z) = xy + 2yz + 2zx - 432 = 0 \rightarrow (1)$

Consider, lagrangian function (or) Auxiliary equation.

$F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$

$F(x,y,z) = xyz + \lambda\,(xy + 2yz + 2zx - 432)$

for maximum or minimum, we have $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

$\times x \qquad \times y \qquad \times z$

$(yz + \lambda(y + 2z) = 0) \rightarrow (2)$; $(xz + \lambda(x + 2z) = 0) \rightarrow (3)$ $(xy + \lambda(2y + 2x) = 0) \rightarrow (7)$

$xyz + \lambda(xy + 2xz) = 0 \rightarrow (4)$ $\quad xyz + \lambda\,(xy + 2yz) = 0 \rightarrow (5)$

$xyz + \lambda\,(2yz + 2xz) = 0 \rightarrow (6)$

$(4) - (5) \Rightarrow \lambda\,(2zx - 2yz) = 0$

$2z\lambda(x - y) = 0$

$x - y = 0$

$\boxed{x = y} \rightarrow (8)$

$(5) - (6) \Rightarrow \lambda(xy - 2xz) = 0$

$\cancel{\lambda} \cancel{x} y = \cancel{\lambda}\, 2 \cancel{x} z$

$y = 2z$

$\boxed{z = \frac{y}{2}} \rightarrow (9)$

sub (8), (9) in (1), $\quad x.x + 2.x.\frac{x}{2} + 2.\frac{x}{2}.x - 432 = 0$

$x^2 + x^2 + x^2 = 432$

$3x^2 = 432$

$x^2 = 144$

$x = 12$

$y = 12$

$z = \frac{12}{2} = 6$

$\therefore\ x = 12,\ y = 12,\ z = 6$

MULTIPLE INTEGRALS

Multiple Integrals is a natural extension of a definite integral to a function of two variables ($\iint_R$) or three variables ($\iiint$) or more variables.

Double integral and triple integral are useful in evaluation of area, volume, mass, centroid and moments of inertia of plane and solid regions.

DOUBLE INTEGRALS:

Let $f(x,y)$ be the function of two variables x, y to evaluate $\iint_R f(x,y)\,dx\,dy$ where R = Region of integration

Case(i): If $x = x_1$ to $x = x_2$ and $y = y_1$ to $y = y_2$ (all limits are constant) be the limits of integration then the order of integration is immaterial.

$$\iint_R f(x,y)\,dx\,dy = \int_{x=x_1}^{x=x_2}\int_{y=y_1}^{y=y_2} f(x,y)\,dy\,dx = \int_{y=y_1}^{y=y_2}\int_{x=x_1}^{x=x_2} f(x,y)\,dx\,dy$$

Case(ii): If $y = y_1(x)$ to $y = y_2(x)$ and $x = x_1$ to $x = x_2$ (x limits are constants) then integration given expression w.r.t. 'y' and then resulting expression with respect to 'x'

$$\iint_R f(x,y)\,dx\,dy = \int_{x=x_1}^{x=x_2}\left[\int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\,dy\right] dx$$

Case (iii): If $x = x_1(y)$ to $x = x_2(y)$ and $y = y_1$ to $y = y_2$ (y limits are constant), then integrate given expression with r.t. 'x' and resulting expression w.r.t. 'y'.

$$\iint_R f(x,y)\,dx\,dy = \int_{y=y_1}^{y=y_2}\left[\int_{x=x_1(y)}^{x=x_2(y)} f(x,y)\,dx\right]dy$$

Evaluate the following

1) $\int_1^2\int_3^4 [xy + e^y]\,dy\,dx$.

Sol:
$$= \int_1^2\left[\int_3^4 [xy+e^y]\,dy\right]dx$$

$$= \int_1^2\left[x\frac{y^2}{2} + e^y\right]_3^4 dx$$

$$= \int_1^2\left[\left(\frac{16}{2}\cdot x + e^4\right) - \left(\frac{9}{2}x + e^3\right)\right]dx$$

$$= \int_1^2\left(\frac{7}{2}x + (e^4 - e^3)\right)dx$$

$$= \left[\frac{7}{2}\frac{x^2}{2} + (e^4-e^3)x\right]_1^2$$

$$= \left(\frac{7}{\cancel{2}}\cdot\frac{\cancel{2^2}}{\cancel{2}} + (e^4-e^3)2\right) - \left(\frac{7}{2}\cdot\frac{1^2}{2} + (e^4-e^3)\right)$$

$$= \left(7 - \frac{7}{4}\right) + (e^4 - e^3)$$

$$= \frac{21}{4} + e^4 - e^3$$

2) $\int_0^5\int_0^{x^2} x(x^2+y^2)\,dx\,dy$.

Sol:
$$= \int_{x=0}^{x=5} \int_{y=0}^{y=x^2} x(x^2+y^2)\, dy\, dx$$

$$= \int_{x=0}^{x=5} \left[\int_{y=0}^{y=x^2} (x^3+xy^2)\, dy\right] dx$$

$$= \int_{x=0}^{5} \cancel{\frac{x^4}{4}} \cancel{+} \left(x^3 y + x.\frac{y^3}{3}\right)_0^{x^2} dx$$

$$= \int_{x=0}^{5} \left(x^3x^2 + x.\frac{x^6}{3} - 0\right) dx$$

$$= \int_{x=0}^{5} \left(x^5 + \frac{x^7}{3}\right) dx$$

$$= \left[\frac{x^6}{6} + \frac{x^8}{24}\right]_0^5$$

$$= \frac{5^6}{6} + \frac{5^8}{24}$$

$$= 5^6\left(\frac{1}{6} + \frac{25}{24}\right)$$

$$= \frac{5^6}{6}\left(1 + \frac{25}{4}\right)$$

$$= \frac{5^6}{6}\left(\frac{29}{4}\right) //$$

(3) $$\int_0^a \int_0^{\frac{y^2}{a}} e^{x/y}\, dx\, dy$$

Sol:
$$\int_{y=0}^{y=a} \left(\int_{x=0}^{x=\frac{y^2}{a}} e^{x/y}\, dx\right) dy$$

$$= \int_0^a \left(\frac{e^{\frac{x}{y}}}{\frac{1}{y}} \right)_0^{y^2/a} dy$$

$$= \int_0^a \left(y\, e^{x/y} \right)_0^{y^2/a} dy$$

$$= \int_0^a \left(y\, e^{\frac{y^2}{ay}} - y\, e^{0/y} \right) dy$$

$$= \int_0^a \left(y\, e^{y/a} - y \right) dy$$

$$= \left(y \cdot \frac{e^{y/a}}{\frac{1}{a}} - 1 \cdot \frac{e^{y/a}}{(\frac{1}{a})^2} - \frac{y^2}{2} \right)_0^a$$

$$= \left(a a e^{a/a} - 1 \cdot \cancel{e^{a/a} . a^2} - \frac{a^2}{2} \right)$$

$$- \left(0 - e^{0/a} . a^2 - 0 \right)$$

$$= -\frac{a^2}{2} + a^2$$

$$= \frac{a^2}{2}$$

-Ex $\int u\,dv$

$$\left(\because \int x^3 e^{ax} dx = x^3 . \frac{e^{ax}}{a} - 3x^2 . \frac{e^{ax}}{a^2} + 6x . \frac{e^{ax}}{a^3} - 6 . \frac{e^{ax}}{a^4} \right)$$

4) $$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$$

Sol: $$\int_{x=0}^{x=1} \left[\int_{y=0}^{y=\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} dy \right] dx$$

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=\sqrt{1+x^2}} \frac{1}{y^2 + (\sqrt{1+x^2})^2} dy \right] dx$$

$$= \int_{x=0}^{x=1} \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}} \right) \right]_{y=0}^{y=\sqrt{1+x^2}} dx$$

$$= \int_{x=0}^{x=1} \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} \right) - \frac{1}{\sqrt{1+x^2}} \tan^{-1}(0) \right] dx$$

$$= \int_{x=0}^{x=1} \left(\frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) - 0 \right) dx.$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$$

$$= \frac{\pi}{4} \left[\log(x + \sqrt{1+x^2}) \right]_0^1$$

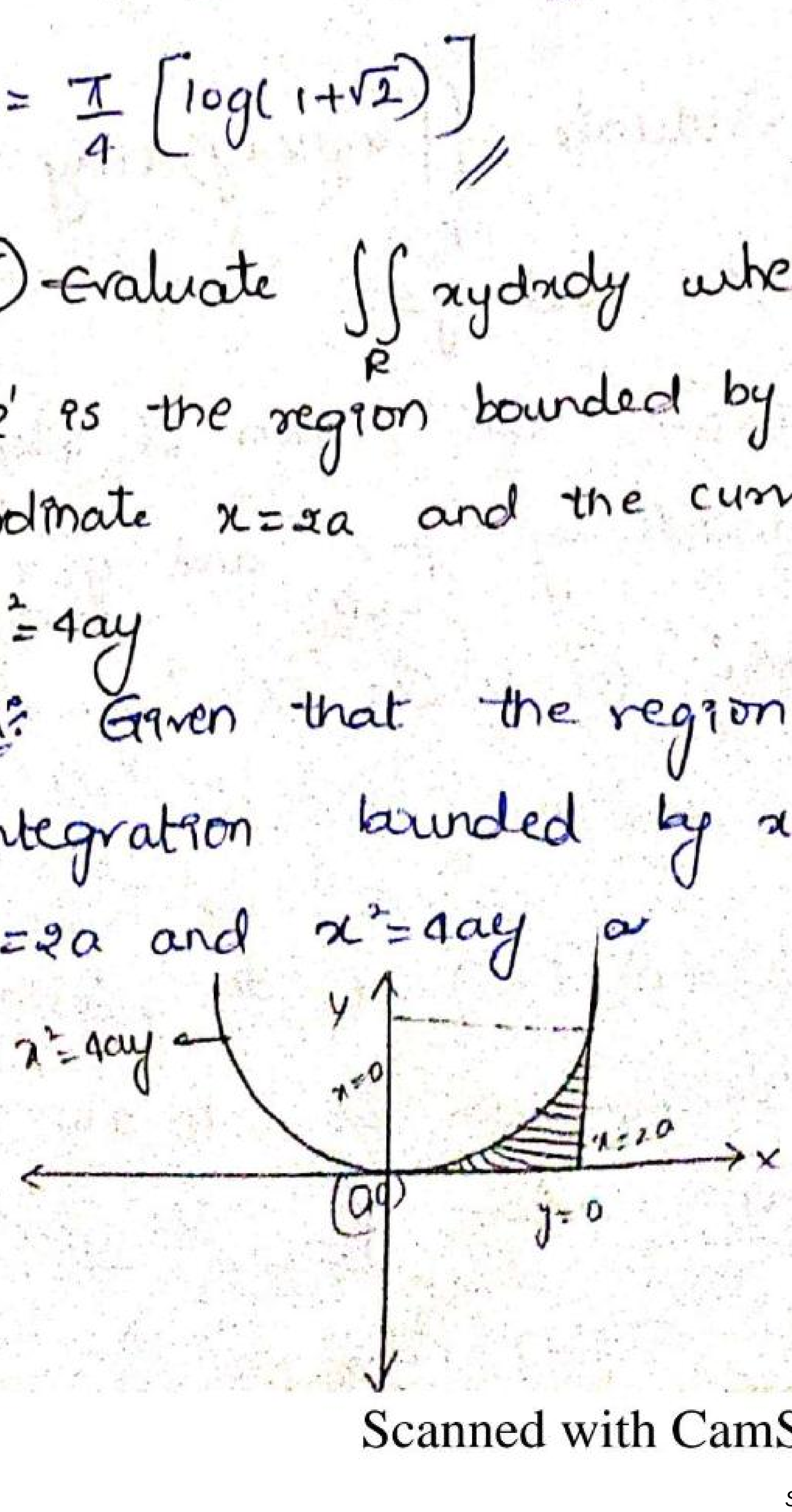
$$= \frac{\pi}{4} \left(\log(1+\sqrt{1+1^2}) - \log(0+\sqrt{1+0}) \right)$$

$$= \frac{\pi}{4} \left(\log(1+\sqrt{2}) - \log 1 \right)$$

$$= \frac{\pi}{4} \left[\log(1+\sqrt{2}) \right]$$

5) Evaluate $\iint_R xy\,dx\,dy$ where 'R' is the region bounded by x-axis, ordinate $x=2a$ and the curve $x^2=4ay$

Sol: Given that the region of integration bounded by x-axis, $x=2a$ and $x^2=4ay$ ar



x - varies from $x=0$ to $x=2a$.

y - varies from $y=0$ to $y=\frac{x^2}{4a}$

$x^2 = 4ay$

$x = 2a$

$(2a)^2 = 4ay$

$4a^2 = 4ay$

$y = a$

$$\therefore \iint_R xy\,dx\,dy = \int_{x=0}^{x=2a} x \left[\int_{y=0}^{y=\frac{x^2}{4a}} y\,dy \right] dx.$$

$$= \int_{x=0}^{x=2a} x \left(\frac{y^2}{2} \right)_0^{x^2/4a} dx$$

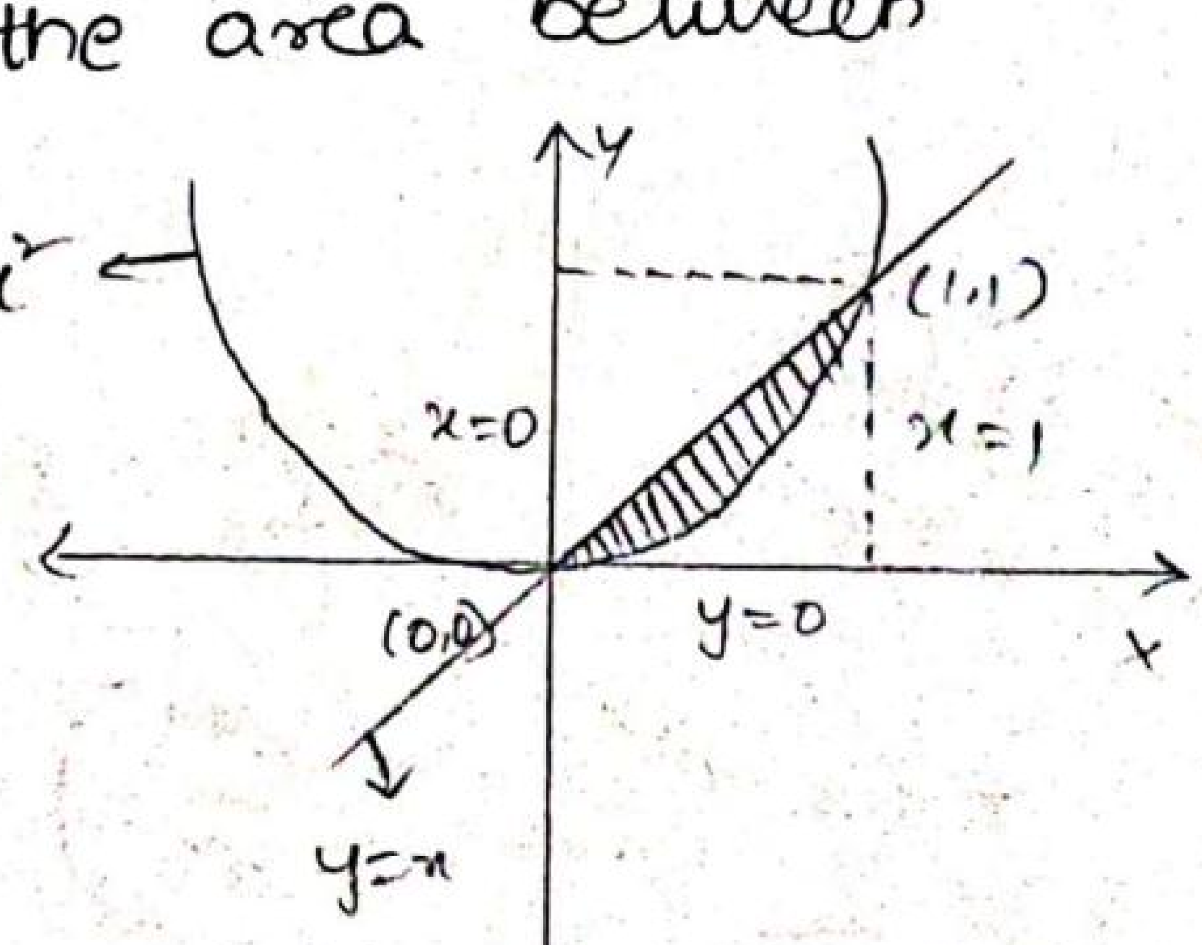
$$= \frac{1}{2} \int_{x=0}^{x=2a} x \left[\frac{x^4}{16a^2} - 0 \right] dx$$

$$\Rightarrow \frac{1}{32a^2} \int_{x=0}^{2a} x^5\,dx = \frac{1}{32a^2} \left(\frac{x^6}{6} \right)_0^{2a}$$

$$= \frac{1}{32a^2} \times \frac{64a^6}{6}$$

$$= \frac{1}{3}a^4 \text{ sq. units}$$

5) Evaluate $\iint_R xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$.



Sol: Given that region of integration bounded by $y=x^2 \to ①$ and $y=x \to ②$

By solving ① & ② i.e., $x = x^2$

$x - x^2 = 0$

$x(1-x) = 0$

$x=0,\ x=1$

For $x=0,\ y=0$

$x=1,\ y=1$

$\therefore$ (0,0), (1,1) are the intersecting points.

x - varies from $x=0$ to $x=1$

y - varies from ~~y=0~~ to $y=x^2$ to $y=x$

$$\Rightarrow \iint_R xy(x+y) = \int_{x=0}^{x=1}\left[\int_{y=x^2}^{y=x}(x^2y+xy^2)\,dy\right]dx$$

$$= \int_{x=0}^{x=1}\left[x^2\frac{y^2}{2}+x\cdot\frac{y^3}{3}\right]_{y=x^2}^{y=x}dx$$

$$= \int_{x=0}^{1}\left[\left(x^2\cdot\frac{x^2}{2}+x\cdot\frac{x^3}{3}\right)-\left(x^2\cdot\frac{x^4}{2}+x\cdot\frac{x^6}{3}\right)\right]dx$$

$$= \int_{x=0}^{x=1}\left[\left(\frac{x^4}{2}+\frac{x^4}{3}\right)-\left(\frac{x^6}{2}+\frac{x^7}{3}\right)\right]dx$$

$$= \int_{x=0}^{x=1}\left(\frac{5x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

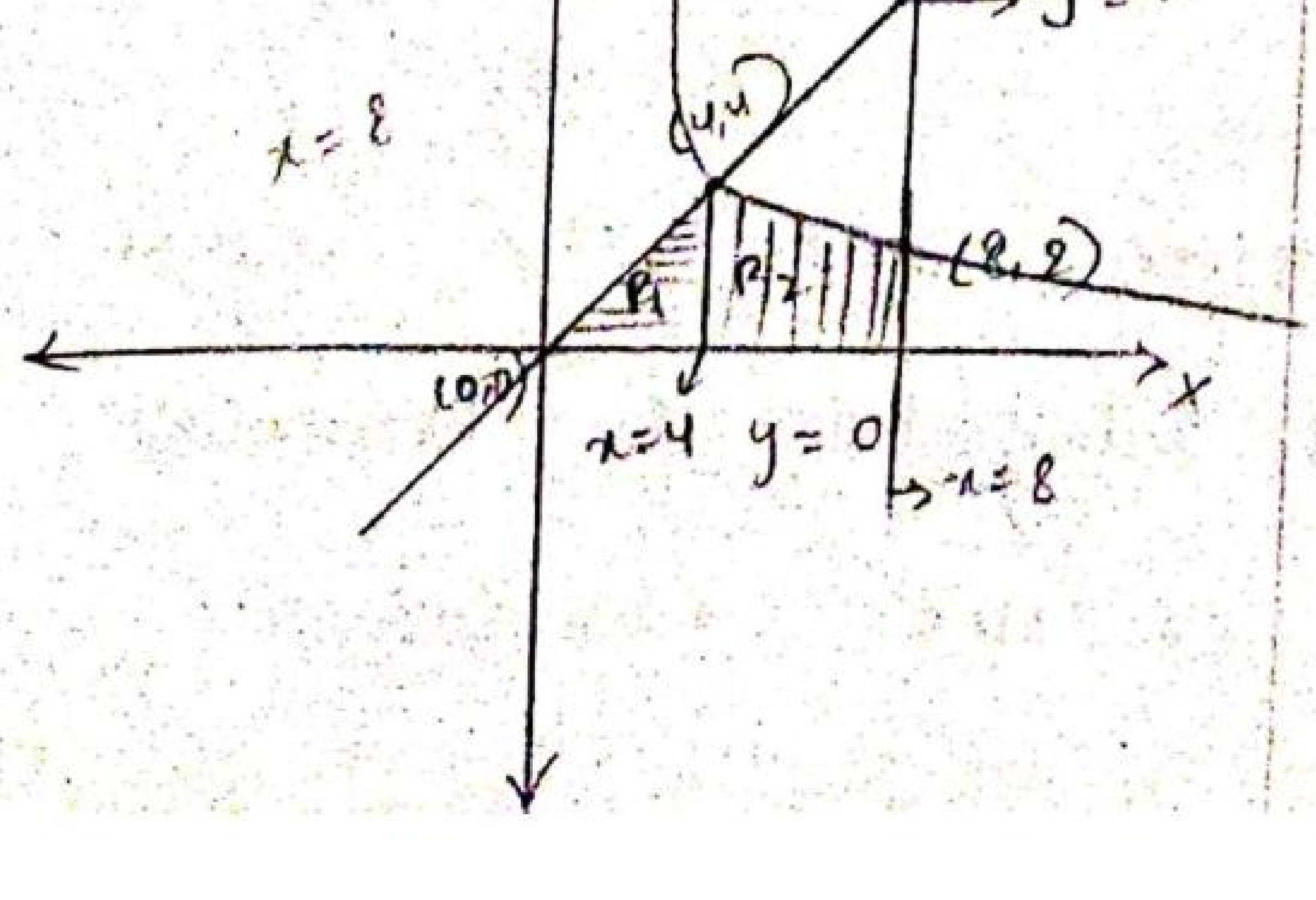
$$= \left[\frac{5}{6}\cdot\frac{x^5}{5}-\frac{1}{2}\cdot\frac{x^7}{7}-\frac{1}{3}\cdot\frac{x^8}{8}\right]_0^1$$

$$= \left(\frac{x^5}{6}-\frac{x^7}{14}-\frac{x^8}{24}\right)_0^1$$

$$= \left[\frac{1}{6}-\frac{1}{14}-\frac{1}{24}-0\right]$$

$$= \frac{3}{56} \text{ Sq units}$$

7) Evaluate ~~doub~~ $\iint_R x^2\,dx\,dy$ where 'R' is the region in first quadrant bounded by the lines $x=y$, $y=0$, $x=8$ and the curve $xy=16$



Given that the region of integration is bounded by $x=y$, $y=0$, $x=8$ and the curve $xy=16$.
Given region splits into 2 regions

R_1	R_2
$x=0$ to $x=4$	$x=4$ to $x=8$
$y=0$ to $y=x$	$y=0$ to $y=16/x$

$$\therefore \iint_R x^2\,dx\,dy = \iint_{R_1} x^2\,dx\,dy + \iint_{R_2} x^2\,dx\,dy$$

$$= \int_{x=0}^{4} x^2\left(\int_{y=0}^{y=x}dy\right)dx + \int_{x=4}^{x=8} x^2\left[\int_{y=0}^{y=16/x}dy\right]dx$$

$$= \int_{x=0}^{x=4} x^2[y]_0^x\,dx + \int_{x=4}^{x=8} x^2\,[y]_0^{16/x}\,dx$$

$$= \int_{x=0}^{4} x^2(x-0)\,dx + \int_{x=4}^{x=8} x^2\left(\frac{16}{x}-0\right)dx$$

$$= \int_{x=0}^{4} x^3\,dx + \int_{x=4}^{x=8} 16x\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^4 + 16\left[\frac{x^2}{2}\right]_4^8$$

$$= \left(\frac{4^4}{4}-0\right) + 16\left(\frac{8^2}{2}-\frac{4^2}{2}\right)$$

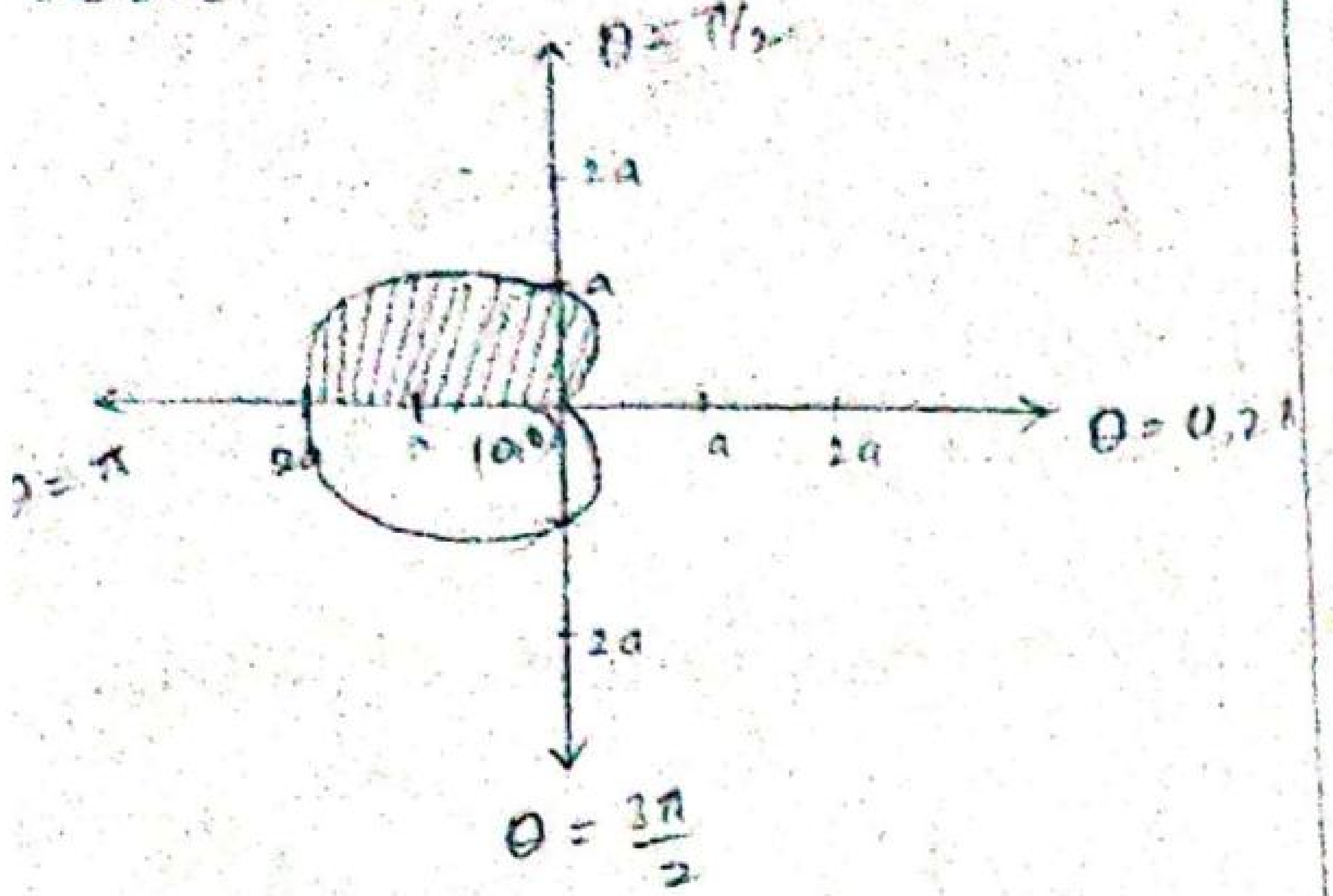
$$= 4^3 + 16\left(\frac{64-16}{2}\right)$$

$$= 64 + 8(48)$$

$$= 384+64$$

$$= 448 \text{ Sq-units.}$$

8) Evaluate $\iint r\sin\theta \cdot dr \cdot d\theta$ over the cardioid $r = a(1-\cos\theta)$ above the initial line.



Given that the region of integration is bounded by $r = a(1-\cos\theta)$ and above the initial line. 'θ' varies from 0 to 'π', 'r' varies from $r=0$ to $r = a(1-\cos\theta)$

$$= \iint r\sin\theta \cdot dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a(1-\cos\theta)} \sin\theta \cdot r \cdot dr \cdot d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \sin\theta \left[\int_{r=0}^{r=a(1-\cos\theta)} r\,dr \right] d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \sin\theta \left(\frac{r^2}{2} \right)_0^{a(1-\cos\theta)} d\theta$$

$$= \int_0^\pi \sin\theta \left(\frac{a^2(1-\cos\theta)^2}{2} - 0 \right) d\theta$$

$$= \frac{a^2}{2} \int_0^\pi (1-\cos\theta)^2 \cdot \sin\theta\,d\theta$$

$$\left(\because \int f(x)^n \cdot f'(x)\,dx = \frac{f(x)^{n+1}}{n+1} + c \right)$$

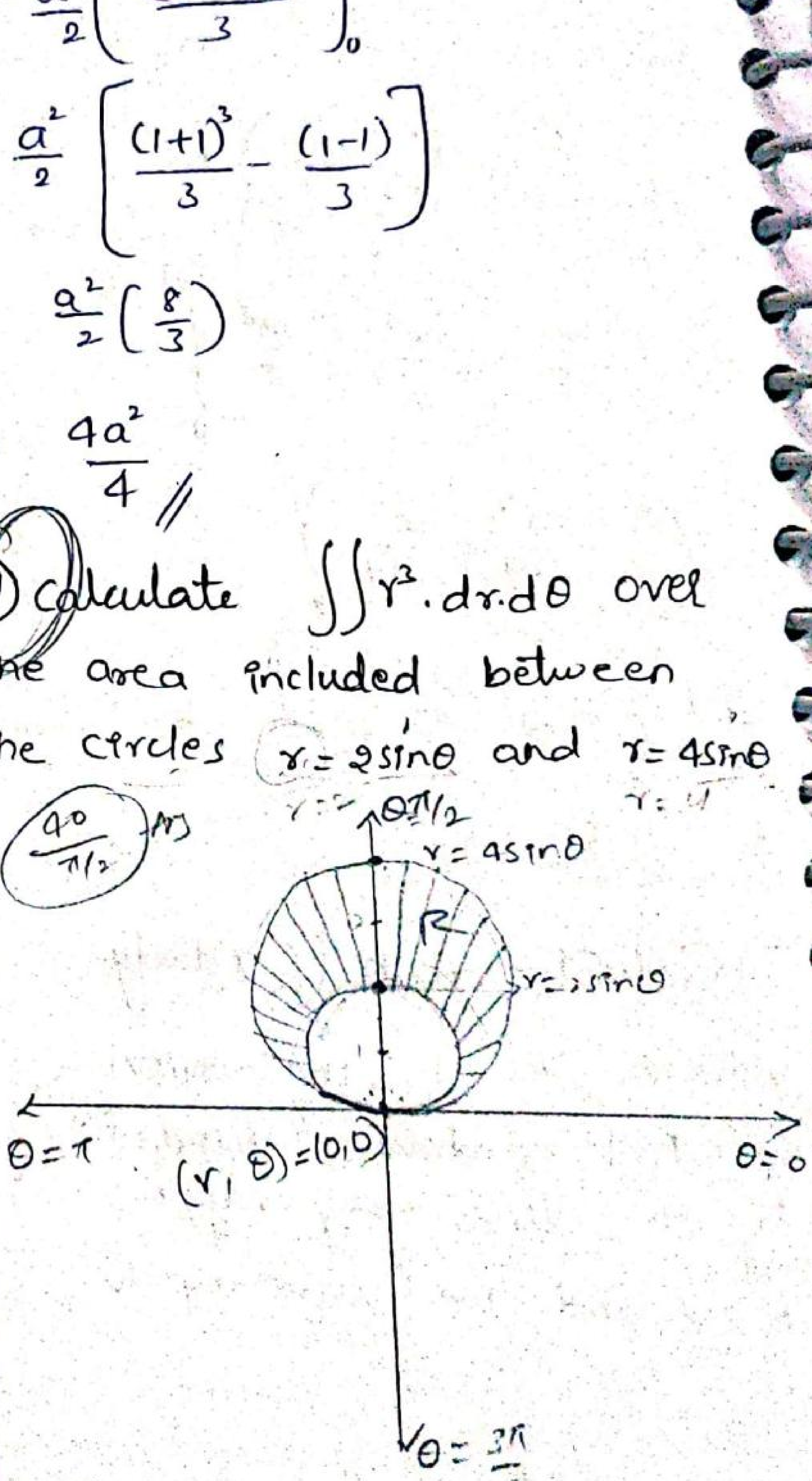
$$= \frac{a^2}{2} \left[\frac{(1-\cos\theta)^3}{3} \right]_0^\pi$$

$$= \frac{a^2}{2} \left[\frac{(1+1)^3}{3} - \frac{(1-1)}{3} \right]$$

$$= \frac{a^2}{2} \left(\frac{8}{3} \right)$$

$$= \frac{4a^2}{4}$$ //

9) Calculate $\iint r^3 \cdot dr \cdot d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$



Given that the region of integration is bounded by $r = 2\sin\theta$, $r = 4\sin\theta$.

∴ 'θ' varies from $\theta = 0$ to $\theta = \pi$ and 'r' varies from $r = 2\sin\theta$ to $r = 4\sin\theta$

$$\iint r^3\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\int_{r=2\sin\theta}^{r=4\sin\theta} r^3\,dr\,d\theta$$

$$= \int_0^\pi \left(\frac{r^4}{4}\right)_{r=2\sin\theta}^{r=4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^\pi \left(256\sin^4\theta - 16\sin^4\theta\right) d\theta$$

$$= \frac{240}{4}\int_0^\pi \sin^4\theta\,d\theta$$

$$= 60 \times 2 \int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= 120 \int_0^{\pi/2} \sin^4\theta\,d\theta$$

$\left(\because \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}\right.$ if 'n' is even,

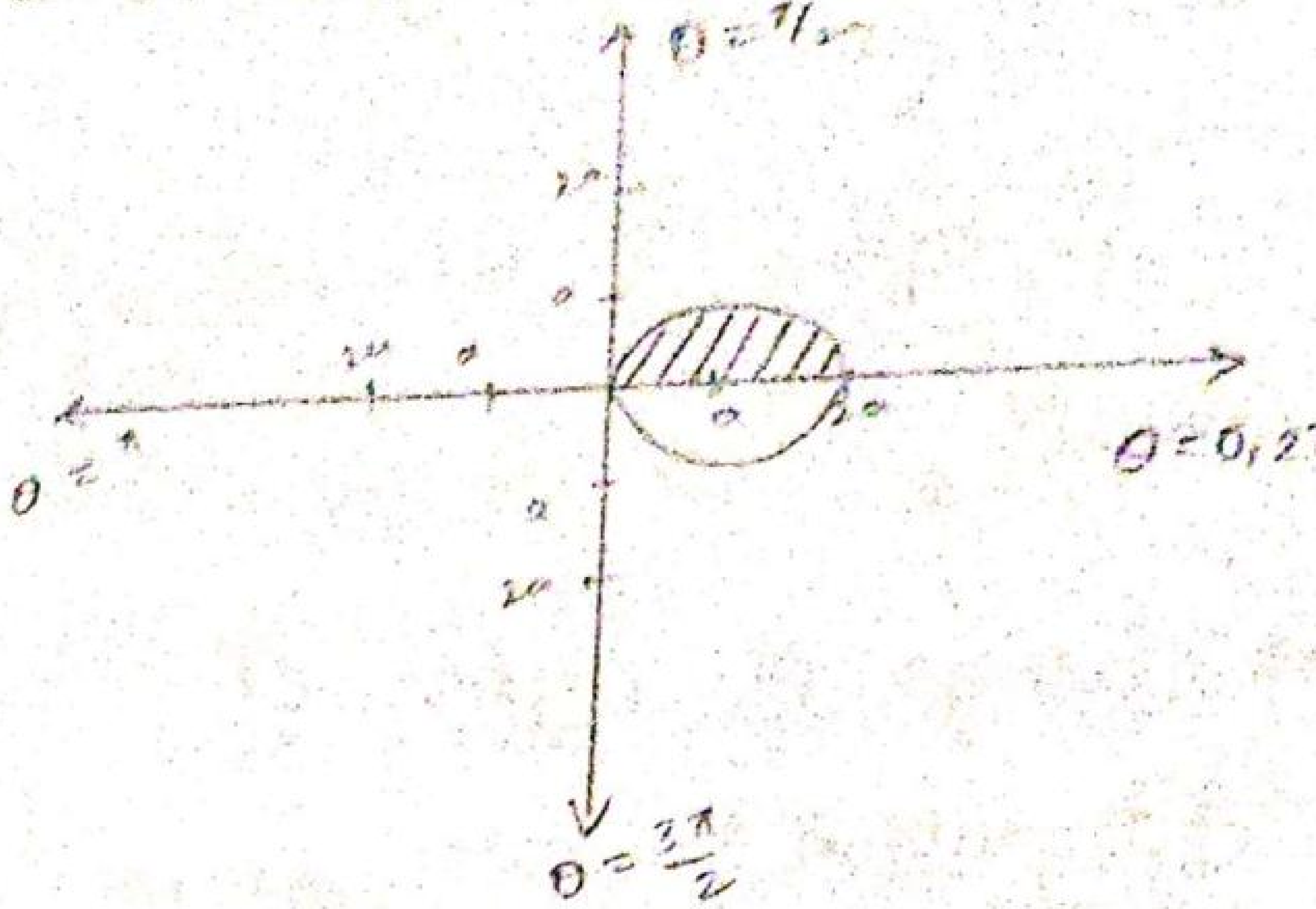
$\left. = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}\right.$ if 'n' is odd $\Big)$

$$= 120\cdot\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}$$

$$= \overset{\overset{15}{\cancel{30}}}{\cancel{120}}\cdot\frac{3}{\cancel{4}}\cdot\frac{1}{\cancel{2}}\cdot\frac{\pi}{2}$$

$$= 45\cdot\frac{\pi}{2}$$

10) Show that $\iint_R r^2\sin\theta\,dr\,d\theta = \frac{2a^3}{3}$ where 'R' is the semi circle where $r = 2a\cos\theta$ above the initial line.



Given that the region of integration is bounded by $r = 2a\cos\theta$, and above the initial line.

∴ 'θ' varies from $\theta = 0$, to $\theta = \frac{\pi}{2}$ and 'r' varies from $r = 0$ to $r = 2a\cos\theta$

$$= \iint r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\int_{r=0}^{r=2a\cos\theta} r^2\,dr\right] d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left(\frac{8a^3\cos^3\theta}{3} - 0\right) d\theta$$

$$= -\frac{8a^3}{3}\int_0^{\pi/2} (\cos\theta)^3(-\sin\theta)\,d\theta$$

$$= -\frac{8a^3}{3}\left[\frac{(\cos\theta)^4}{4}\right]_0^{\pi/2}$$

$$= \frac{-8a^3}{3}\left[\frac{(\cos\pi/2)^4}{4} - \frac{(\cos 0)^4}{4}\right]$$

$$= -\frac{8a^3}{3}\left(0 - \frac{1}{4}\right)$$

$$= \frac{8a^3}{3} \times \frac{1}{4}$$

$$= \frac{2a^3}{3}$$

⇒ Change of order of Integration :

$$\boxed{\int_{x=a}^{x=b}\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\,dx = \int_{y=c}^{y=d}\int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)\,dx\,dy}$$

(*) Evaluate $\displaystyle\int_{x=a}^{x=b}\int_{y=\phi_1(x)}^{y_2=\phi_2(x)} f(x,y)\,dy\,dx$

by change of order of integration, we must change the limits for fixed y i.e., $y=c$ to $y=d$ and x varies from $x=\phi_1(y)$ to $x=\phi_2(y)$. i.e.

$$\int_{x=a}^{x=b}\int_{y=\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx = \int_{y=c}^{y=d}\int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)\,dx\,dy.$$

) By change of order of ~~order~~ of integration evaluate

$$\int_{x=0}^{x=4a}\int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dy\,dx.$$

Sol: Given that the region of integration bounded by $x=0$ to $x=4a$ and $y=\frac{x^2}{4a}$ to $y=2\sqrt{ax}$.
$(x^2=4ay)$
$(y^2=4ax)$

Here $x^2 = 4ay$

$x^2 = 4a\,2\sqrt{ax}$

$x^4 = 64a^3 . x$

$x^4 - 64a^3x = 0$

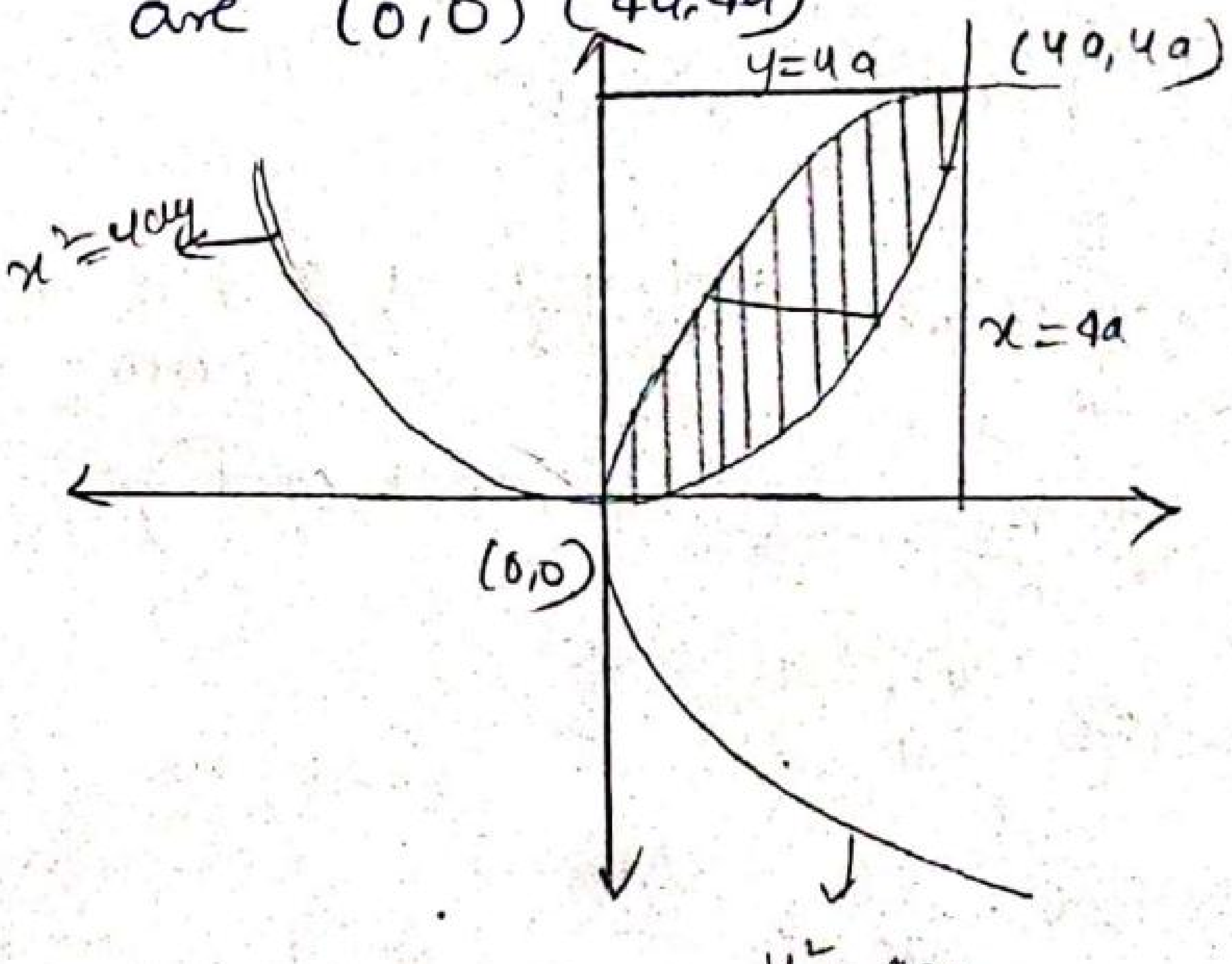
$x(x^3 - 64a^3) = 0$

$x = 0,\ x = 4a$

when $x=0,\ y=0$

$x = 4a,\ y = 4a$

∴ Intersection points of $x^2 = 4ay$ and $y^2 = 4ax$ are $(0,0)$ $(4a,4a)$



$$\therefore \int_{x=0}^{x=4a}\int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dy\,dx = \int_{y=0}^{y=4a}\int_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}} dx\,dy$$

$$\Rightarrow \int_{y=0}^{4a}[x]_{\frac{y^2}{4a}}^{2\sqrt{ay}}\,dy = \int_{y=0}^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= \left[2\sqrt{a}x \cdot y^{3/2} \cdot \frac{2}{3} - \frac{1}{4a} \cdot \frac{y^3}{3}\right]_0^{4a}$$

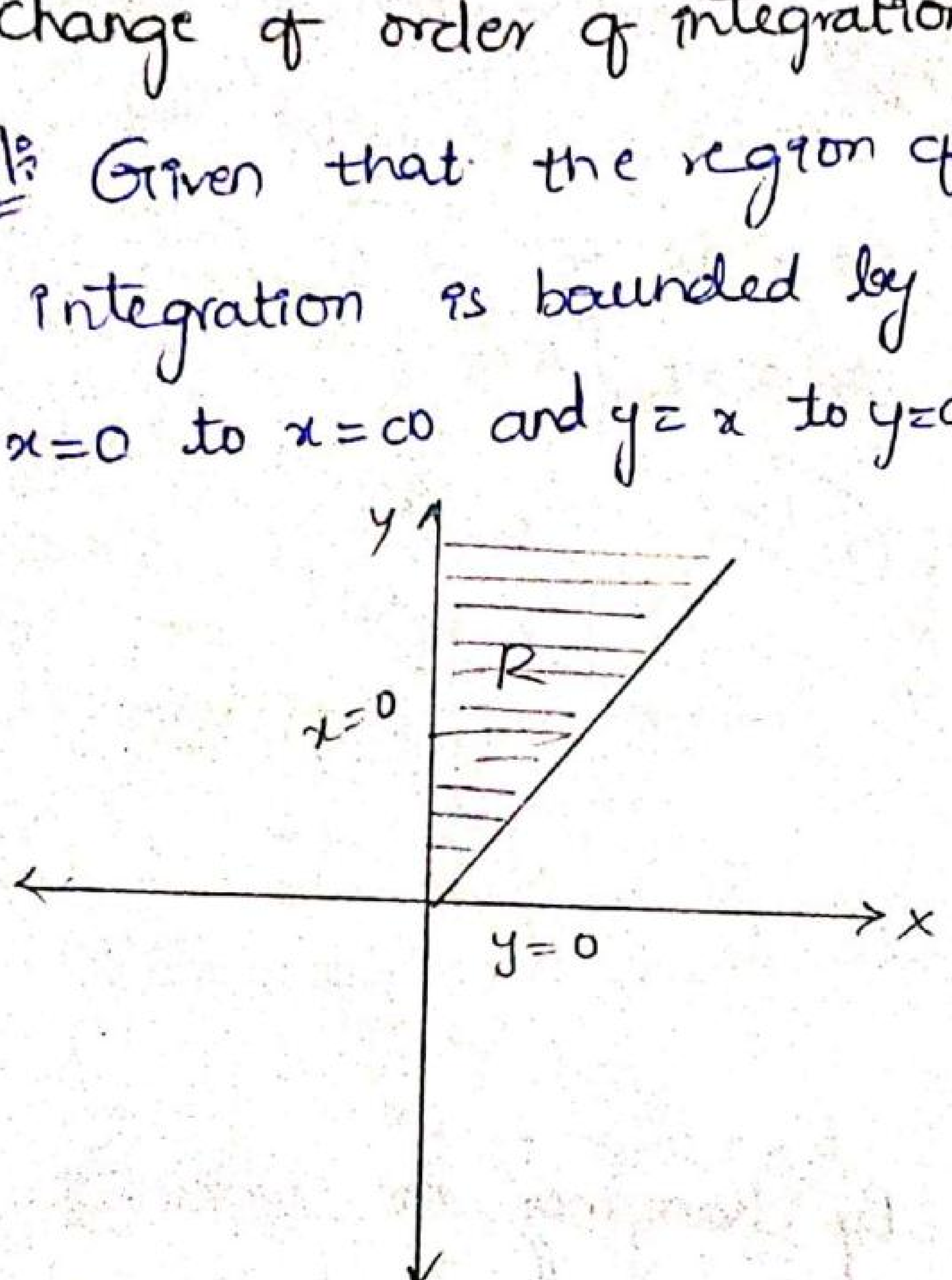
$$= \left(\frac{4}{3}\sqrt{a}(4a)^{3/2} - \frac{1}{12a}(4a)^3\right) - (0-0)$$

$$= \frac{32a^2}{3} - \frac{64}{12}a^2$$

$$= \frac{16}{3}a^2.$$

2) Evaluate $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$ by change of order of integration

Sol: Given that the region of integration is bounded by $x=0$ to $x=\infty$ and $y=x$ to $y=\infty$



$$\therefore \int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dy\,dx = \int_{y=0}^{y=\infty}\int_{x=0}^{x=y} \frac{e^{-y}}{y}\,dx\,dy$$

$$\Rightarrow \int_{y=0}^{\infty} \frac{e^{-y}}{y}\left[\int_{x=0}^{x=y} dx\right]dy = \int_{y=0} \frac{e^{-y}}{y}(x)_0^y\,dy$$

$$\Rightarrow \int_{y=0}^{\infty} \frac{e^{-y}}{y}(y)\,dy = \int_{y=0}^{\infty} e^{-y}\,dy$$

$$= \left(-e^{-y}\right)_0^{\infty}$$

$$= -e^{-\infty} + e^0$$

$$= 0+1$$

$$= 1$$

3) Evaluate $\int_0^1\int_{e^x}^{e} \frac{dy\,dx}{\log y}$ by changing the order of integration

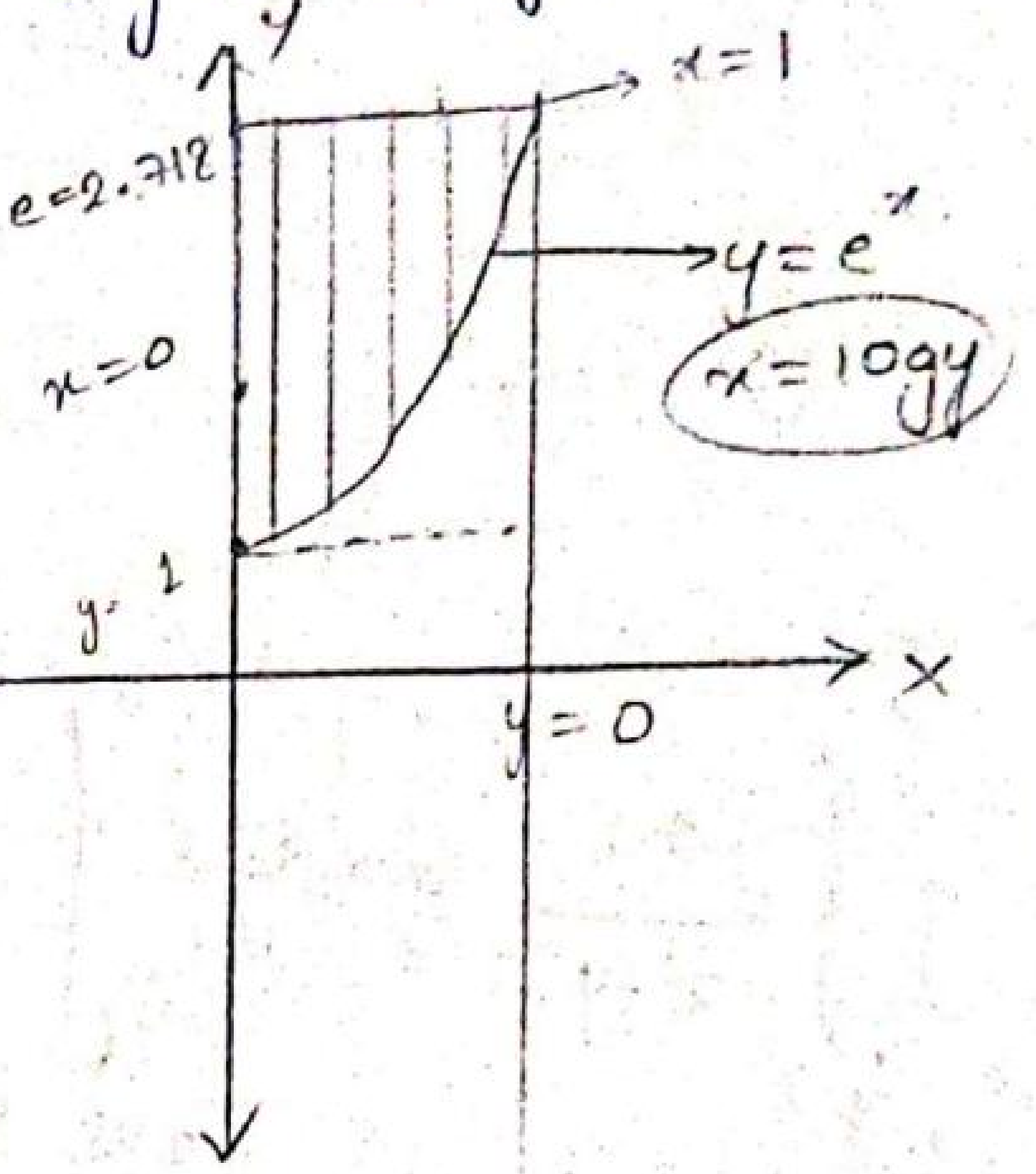
Sol: Given that the region of integration is bounded by $x=0$ to $x=1$ and $y=e^x$ to $y=e=2.718$

$y=e^x$

$x=0,\ y=1$

$x=0.1,\ y=1.12$

$x=0.2,\ y=1.23$



$$\int_{x=0}^{x=1}\int_{y=e^x}^{y=e} \frac{1}{\log y}\,dy\,dx = \int_{y=1}^{e}\int_{x=0}^{\log y} \frac{1}{\log y}\,dx\,dy$$

$$= \int_{y=1}^{e} \frac{1}{\log y}\left[x\right]_0^{\log y} dy$$

$$= \int_{y=1}^{e} \frac{1}{\log y}(\log y - 0)\,dy$$

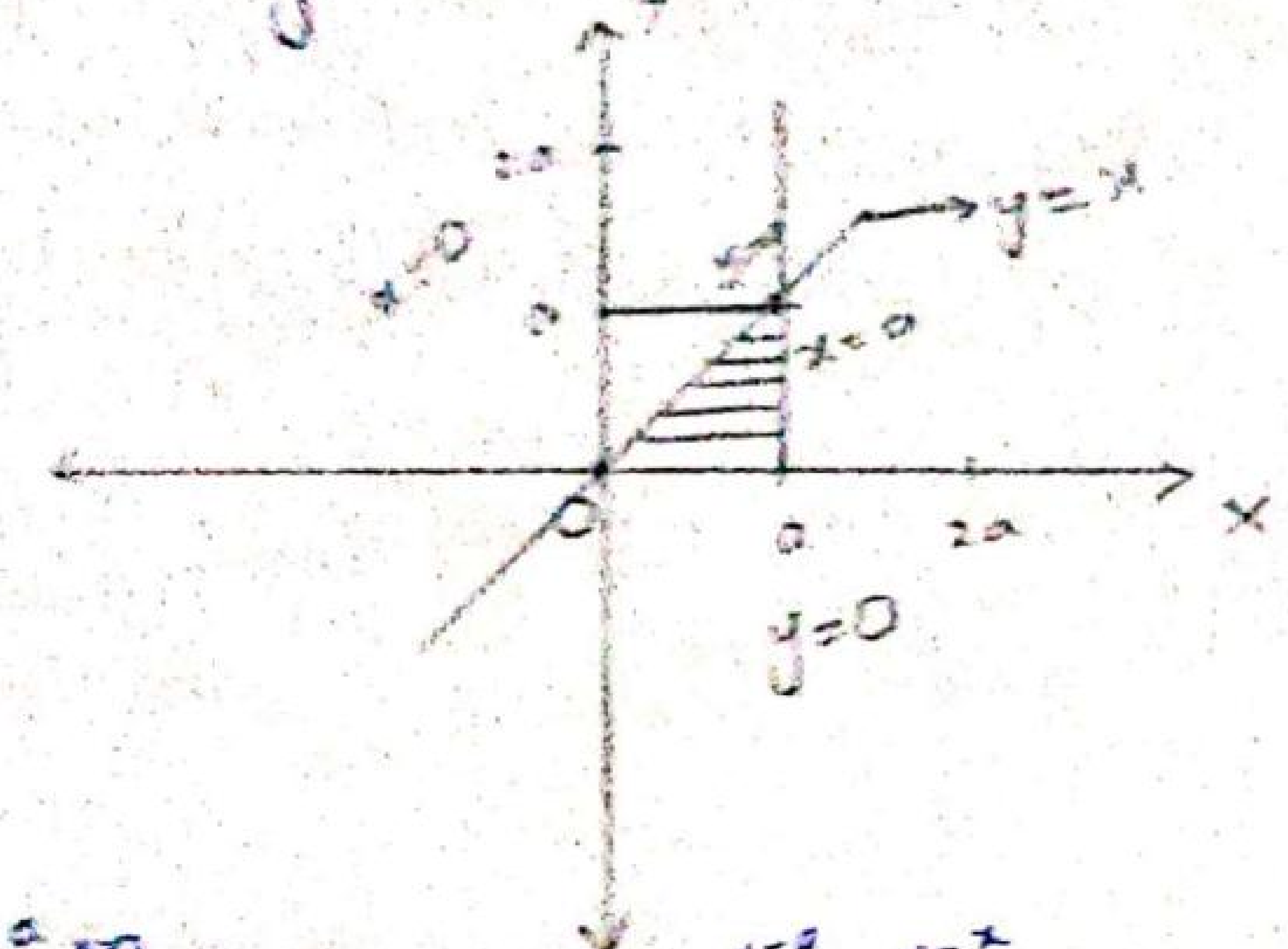
$$= \int_{y=1}^{e} dy$$

$$= [y]_1^e$$

$= e - 1$

4) Evaluate $\int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ by change of order of integration

Sol: Given region of integration is bounded by $y=0$ to $y=a$ and $x=y$ to $x=a$



$$\int_{y=0}^{a}\int_{x=y}^{x=a} \frac{x}{x^2+y^2}\,dx\,dy = \int_{x=0}^{x=a}\int_{y=0}^{y=x}\frac{x}{x^2+y^2}\,dy\,dx$$

$$=\int_{x=0}^{x=a} x\left(\int_0^x \frac{1}{x^2+y^2}\,dy\right)dx.$$

$$=\int_{x=0}^{a} x\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$=\int_{x=0}^{a}\left[\tan^{-1}\left(\frac{x}{x}\right)-\tan^{-1}\left(\frac{0}{x}\right)\right]dx$$

$$=\int_{x=0}^{a}\tan^{-1}(1)\,dx$$

$$=\frac{\pi}{4}\int_0^a dx$$

$$=\frac{\pi}{4}(x)_0^a$$

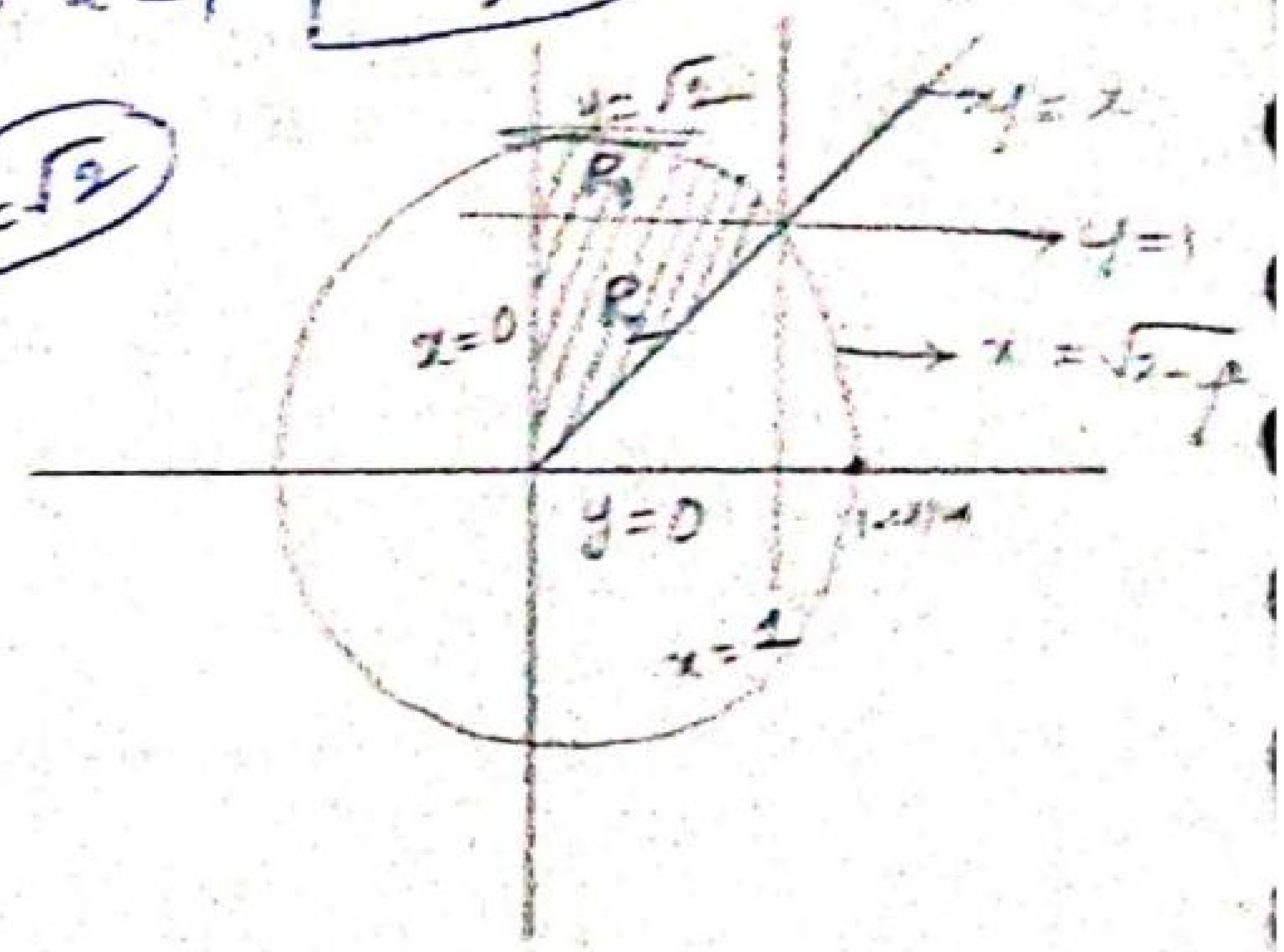
$$=\frac{\pi}{4}(a-0)$$

$$=\frac{\pi a}{4}$$

5) Evaluate $\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing the order of integration.

Sol: Given region of integration is bounded by $y=x$ to $y=\sqrt{2-x^2}$ and $x=0$ to $x=1$ $\boxed{x^2+y^2=2}$ $\boxed{x^2+y^2=r^2}$

$r=\sqrt{2}$



By changing the order of integration

R_1: $y=0$ to $y=1$ & $x=0$ to $x=y$

R_2: $y=1$ to $y=\sqrt{2}$ & $x=0$ to $x=\sqrt{2-y^2}$

$$\therefore \int_0^1\int_x^{\sqrt{2-x^2}}\frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_{y=0}^{y=1}\left[\int_{x=0}^{x=y}\frac{x\,dx}{\sqrt{x^2+y^2}}\right]dy + \int_{y=1}^{y=\sqrt{2}}\left[\int_{x=0}^{x=\sqrt{2-y^2}}\frac{x\,dx}{\sqrt{x^2+y^2}}\right]dy$$

$$= \frac{1}{2}\int_{y=0}^{y=1}\left[\int_{x=0}^{x=y}\frac{2x\,dx}{\sqrt{x^2+y^2}}\right]dy + \frac{1}{2}\int_{y=1}^{y=\sqrt{2}}\left[\int_{x=0}^{x=\sqrt{2-y^2}}\frac{2x\,dx}{\sqrt{x^2+y^2}}\right]dy$$

$$= \frac{1}{2}\int_{y=0}^{y=1}\left[2\sqrt{x^2+y^2}\right]_{x=0}^{x=y}dx + \frac{1}{2}\int_{y=1}^{y=\sqrt{2}}\left[2\sqrt{x^2+y^2}\right]_{0}^{\sqrt{2-y^2}}dy$$

$$= \int_{y=0}^{y=1}\left(\sqrt{2y^2}-\sqrt{y^2}\right)dy + \int_{y=1}^{\sqrt{2}}\left(\sqrt{2}-\sqrt{y^2}\right)dy$$

$$= \int_{y=0}^{1}(\sqrt{2}-1)\,y\,dy + \int_{y=1}^{\sqrt{2}}(\sqrt{2}-y)\,dy$$

$$= (\sqrt{2}-1)\left(\frac{y^2}{2}\right)_0^1 + \left(\sqrt{2}y-\frac{y^2}{2}\right)_1^{\sqrt{2}}$$

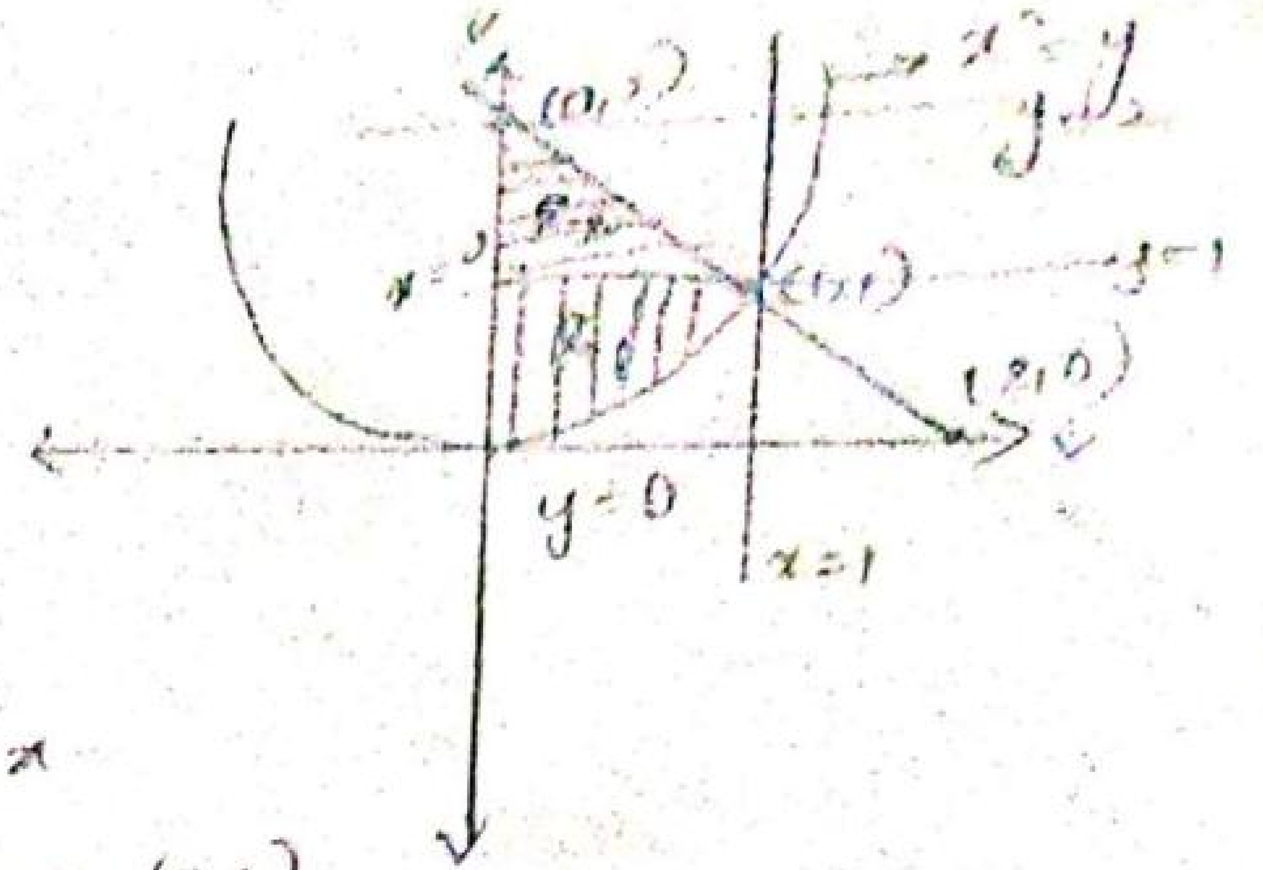
$$= (\sqrt{2}-1)\left(\frac{1}{2}\right) + (\sqrt{2}.\sqrt{2}-1) - \left(\sqrt{2}-\frac{1}{2}\right)$$

$$= \cancel{\frac{1}{\sqrt{2}}} - \frac{1}{2} + (2-1) - \cancel{\sqrt{2}} + \cancel{\frac{1}{2}}$$

$$= 1 - \frac{1}{\sqrt{2}}$$

6) Evaluate $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ by change of order of integration.

Sol: Given region of integration is bounded by $y=x^2$ to $y=2-x$ and $x=0$ to $x=1$



$y = 2-x$

$x=0, y=2$ (0,2)

$y=0, x=2$ (2,0)

By changing the order of integration

R_1: $y=0$ to $y=1$, $x=0$ to $x=\sqrt{y}$

R_2: $x=0$ to $x=2-y$, $y=1$ to $y=2$

$$\therefore \int_0^1\int_{x^2}^{2-x} xy\,dy\,dx = \int_0^1\left[y\int_0^{\sqrt{y}} x\,dx\right]dy + \int_{y=1}^{2} y\left[\int_{x=0}^{x=2-y} x\,dx\right]dy$$

$$= \int_{y=0}^{y=1} y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}}dy + \int_{y=1}^{2} y\left(\frac{x^2}{2}\right)_0^{2-y}dy$$

$$= \int_{y=0}^{1}\frac{y}{2}\left(y-0\right)dy + \int_{y=1}^{2}\left(\frac{y}{2}\right)\left(4+y^2-4y\right)-0\,dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2\left(y^3-4y^2+4y\right)dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\left(\frac{y^4}{4}-\frac{4y^3}{3}+\frac{4y^2}{2}\right)_1^2$$

$$= \frac{1}{6}(1-0) + \frac{1}{2}\left[\left(\frac{16}{4}-\frac{32}{3}+8\right) - \left(\frac{1}{4}-\frac{4}{3}+2\right)\right]$$

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$$= \frac{1}{6} + \frac{1}{2}\left[\left(\frac{12-32+24}{3}\right) - \left(\frac{3-16+24}{12}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(\frac{4}{3}\right) - \left(\frac{11}{12}\right)\right]$$

$$= \frac{1}{6} + \frac{4}{6} - \frac{11}{24}$$

$$= \frac{5}{6} - \frac{11}{24}$$

$$= \frac{1}{6}\left(5 - \frac{1}{4}\right)$$

$$= \frac{1}{6}\left(\frac{19}{4}\right)$$

$$= \frac{19}{24}$$

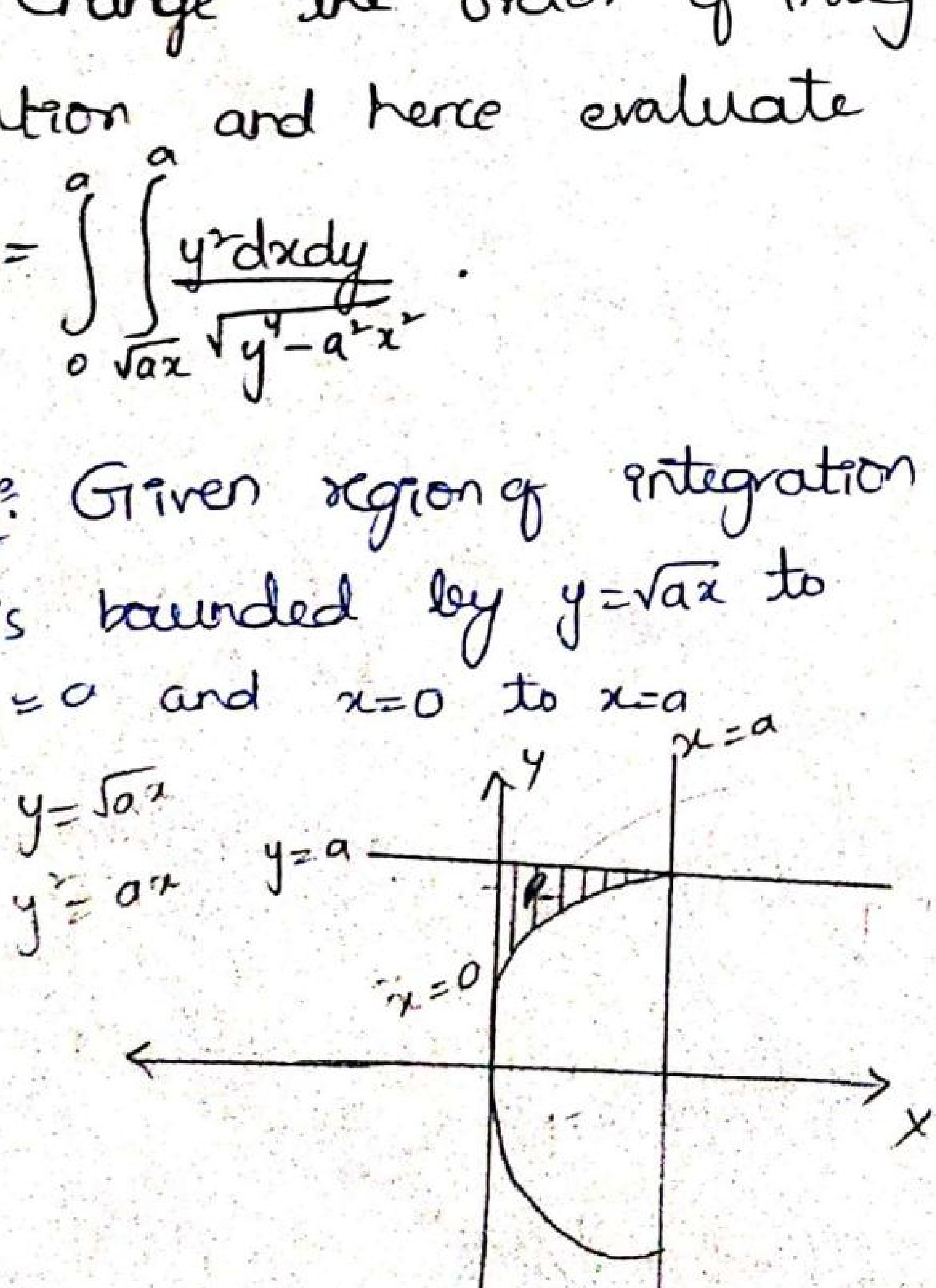
$$= \frac{1}{6} + \frac{1}{2}\left[\left(12 - \frac{32}{3}\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$= \frac{3}{8}$$

7) Change the order of integration and hence evaluate

$$I = \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dx\,dy}{\sqrt{y^4 - a^2x^2}}.$$

Sol: Given region of integration is bounded by $y = \sqrt{ax}$ to $y = a$ and $x = 0$ to $x = a$



By the change of order of integration

$$\int_{x=0}^{a} \int_{y=\sqrt{ax}}^{a} \frac{y^2\,dx\,dy}{\sqrt{y^4 - a^2x^2}} = \int_{y=0}^{a} \int_{x=0}^{y^2/a} \frac{y^2}{\sqrt{y^4 - a^2x^2}}\,dx\,dy$$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{y^2/a} \frac{y^2}{\sqrt{y^4 - a^2x^2}}\,dx\right] dy$$

$$= \int_{y=0}^{a} y^2 \left[\int_{x=0}^{y^2/a} \frac{1}{\sqrt{y^4 - a^2x^2}}\,dx\right] dy$$

$$= \frac{1}{a}\int_{y=0}^{a} y^2 \left[\int_{x=0}^{x=y^2/a} \frac{1}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}\,dx\right] dy$$

$$= \frac{1}{a}\int_{y=0}^{a} y^2 \cdot \left[\sin^{-1}\left(\frac{x}{\left(\frac{y^2}{a}\right)}\right)\right]_0^{y^2/a} dy$$

$$= \frac{1}{a}\int_0^a y^2\left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy$$

$$= \frac{1}{a}\int_0^a y^2 \cdot \frac{\pi}{2}\,dy$$

$$= \frac{\pi}{2a}\int_0^a y^2\,dy$$

$$= \frac{\pi}{2a}\left.\frac{y^3}{3}\right|_0^a$$

$$= \frac{\pi}{2a}\left(\frac{a^3}{3} - 0\right)$$

$$\Rightarrow \frac{\pi}{2\cancel{a}}\left(\frac{a^{\cancel{3}\,2}}{3}\right) = \frac{2\pi a^2}{6}$$

⇒ Change of Variables in double integrals:

Let x, y be the variables changed to the new variables u, v by the transformation $x = \phi(u,v)$ and $y = \psi(u,v)$ then

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R_{uv}} f(\phi(u,v), \psi(u,v))\,|J|\,du\,dv$$

where $|J| = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$ is the Jacobian of x, y w.r.t. u, v.

To change ~~partition~~ cartesian coordinates (x,y) to polar coordinates (r,θ):

we have $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2+y^2 = r^2$ then

$$|J| = r = \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$\therefore\ dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta.$$

$$\therefore \iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r,\theta)\,|J|\,dr\,d\theta$$

$$= \iint_{R'} f(r,\theta)\,r\,dr\,d\theta.$$

Ex: By using the transformation $x+y = u$, $y = uv$, show that

$$\int_0^1\int_0^{1-x} e^{y/x+y}\,dy\,dx = \frac{1}{2}(e-1)$$

Sol: Given, $x+y = u \rightarrow (1)$

$y = uv \rightarrow (2)$

From $(1) \Rightarrow x+y = u$

$\Rightarrow x + uv = u$ ($\because$ from (2))

$\Rightarrow x = u - uv$

$\Rightarrow x = u(1-v)$

If $x = u(1-v)$, $y = uv$

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$|J| = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$|J| = u - \cancel{uv} + \cancel{uv}$$

$$|J| = u.$$

$\therefore\ dx\,dy = |J|\,du\,dv$

$dx\,dy = u.du\,dv$

The region 'R' in xy-plane ~~becomes~~ is bounded by the lines $x=0$ to $x=1$ and $y=0$ to $y=1-x$.

Now, $x=0 \Rightarrow u(1-v) = 0$

$u=0$, $v=1$

$y=0 \Rightarrow uv = 0$

$u=0$, $v=0$

$y = 1-x \Rightarrow u\cancel{v} = 1-u+u\cancel{v}$

$u=1$

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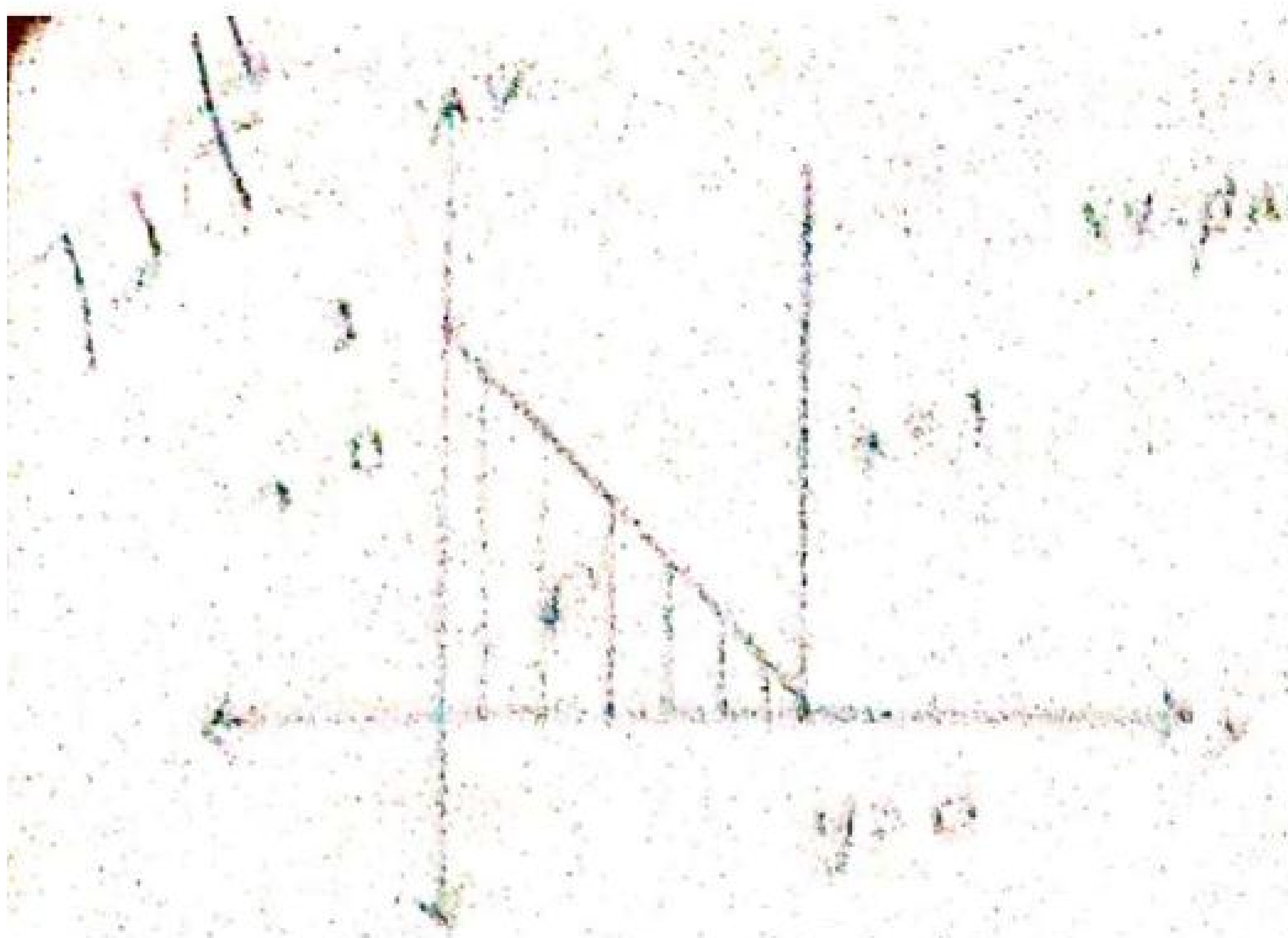
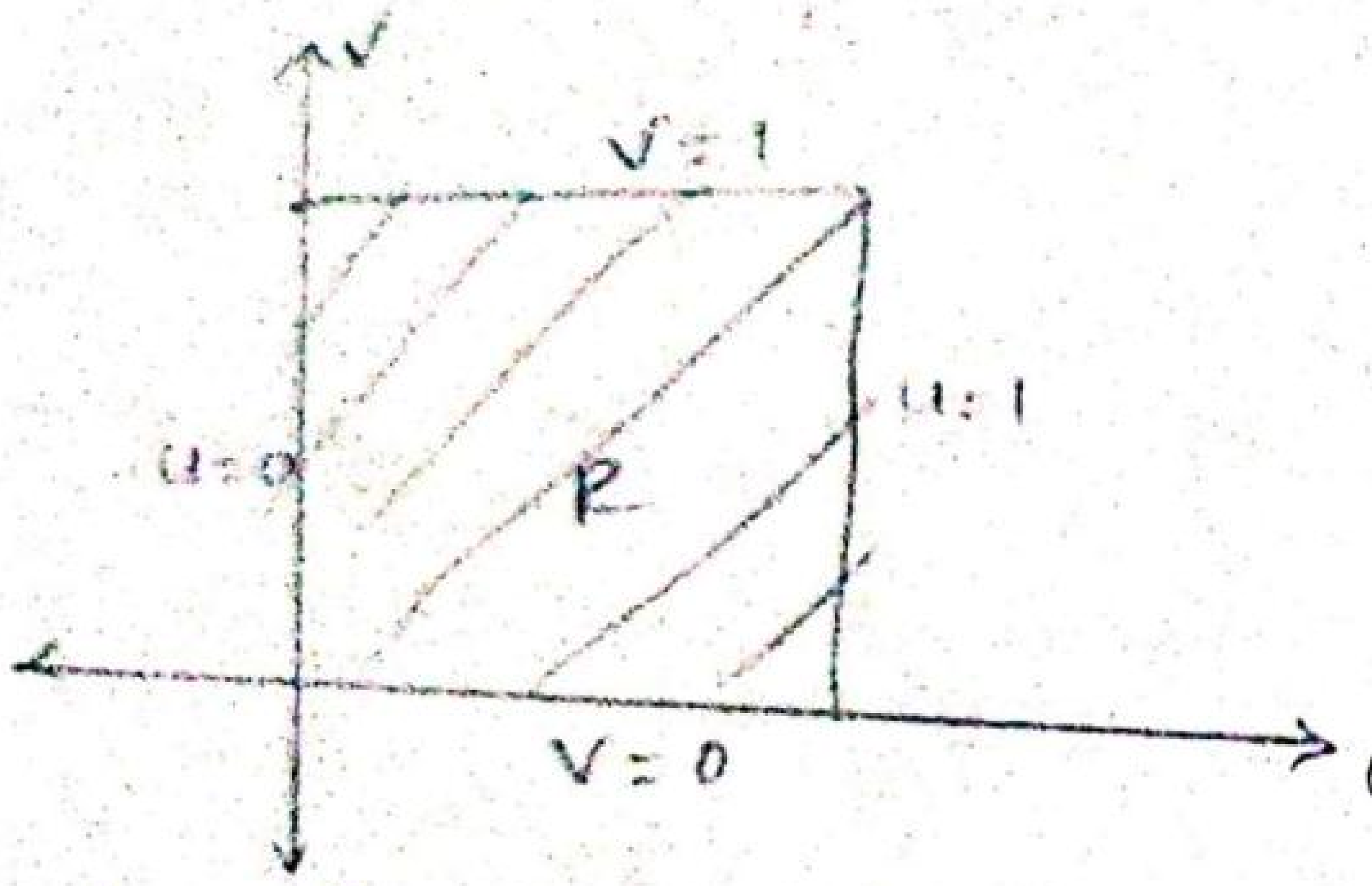


The region R in the xy-plane becomes the region R' in uv plane which is a square bounded by the lines $u=0$, $u=1$, $v=0$ and $v=1$



$$\therefore \int_0^1\int_0^{1-x} e^{y/x+y}\,dy\,dx = \int_{u=0}^{1}\int_{v=0}^{1} e^{\frac{uv}{u}}\,|J|\,du\,dv$$

$$= \int_{u=0}^{1}\int_{v=0}^{1} e^{v}\,u\,du\,dv$$

$$= \int_{u=0}^{1} u\,du \cdot \int_{v=0}^{1} e^{v}\,dv$$

$$= \left(\frac{u^2}{2}\right)_0^1 \cdot \left(e^{v}\right)_0^1$$

$$= \left(\frac{1}{2}-0\right)\left(e^{1}-e^{0}\right)$$

$$= \frac{1}{2}(e-1)$$

*1) Evaluate $\iint_D xy\sqrt{1-x-y}\,dx\,dy$ where 'D' is the region bounded by the lines $x=0$, $y=0$ and $x+y=1$ using transformation $x+y=u$, $y=uv$

Sol: Given $x+y=u \rightarrow (1)$

$y=uv \rightarrow (2)$

$(1) \Rightarrow x+y=u$

$x+uv=u$

$\therefore\; x=u(1-v)$, $y=uv$

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$|J| = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$|J| = u - uv + uv$$

$$|J| = u$$

$\therefore\; dx\,dy = |J|\,du\,dv$

$dx\,dy = u \cdot du \cdot dv.$

$\therefore$ 'D' is the region of integration bounded by the lines $x=0$, $y=0$ and $x+y=1$.

Now, $x=0 \Rightarrow u(1-v)=0$

$u=0,\ v=1$

$y=0 \Rightarrow uv=0$

$u=0,\ v=0$

$x+y=1 \Rightarrow u-\cancel{uv}+\cancel{uv}=1$

$u=1$

∴ The region 'D'' bounded by the lines $u=0, u=1, v=0, v=1$

$$\therefore \iint_D xy\sqrt{1-x-y}\,dx\,dy = \int_{u=0}^{1}\int_{v=0}^{1} u(1-u)\cdot uv \cdot \sqrt{1-u+uv-uv}\,|J|\,du\,dv$$

$$= \int_{u=0}^{1}\int_{v=0}^{1} u^2(v-v^2)\sqrt{1-u}\;u\,du\,dv$$

$$= \int_{u=0}^{1}\int_{v=0}^{1} u^3(v-v^2)(1-u)^{1/2}\,dv\,du$$

$$= \int_{u=0}^{1} u^3(1-u)^{1/2}\,du \cdot \int_{v=0}^{1}(v-v^2)\,dv$$

$$= \int_{u=0}^{1} u^{4-1}(1-u)^{3/2-1}\,du \times \left(\frac{v^2}{2}-\frac{v^3}{3}\right)_0^1$$

$$= \beta\left(4,\frac{3}{2}\right)\cdot\left[\left(\frac{1}{2}-\frac{1}{3}\right)-0\right]$$

$$= \frac{1}{6}\cdot\frac{\Gamma(4)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(4+\frac{3}{2}\right)}$$

$$\left(\because \int_0^1 x^{m-1}\cdot(1-x)^{n-1}\cdot dx = \beta(m,n)\right)$$

$$\left(\because \beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}\right)$$

$$= \frac{1}{6}\times\frac{3!\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{16}{945}$$

$$\left(\because \Gamma(n) = (n-1)!\right)$$

$$\left(\because \Gamma(n) = (n-1)\Gamma(n-1)\right)$$

2) Evaluate $\iint_R (x+y)^2\,dx\,dy$ where 'R' is the parallelogram in the xy-plane with the vertices (1,0), (3,1), (2,2) and (0,1) by using the transformations $u = x+y$ and $v = x-2y$

Sol: Given, $u = x+y \rightarrow (1)$

$v = x-2y \rightarrow (2)$

From (1) and (2) $\Rightarrow u-v = 3y$

$y = \frac{1}{3}(u-v)$

$(1) \Rightarrow u = x + \frac{1}{3}(u-v)$

$x = u - \frac{1}{3}(u-v)$

$x = \frac{3u-u+v}{3}$

$x = \frac{1}{3}(2u+v)$

$\therefore x = \frac{1}{3}(2u+v)$ and $y = \frac{1}{3}(u-v)$

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$|J| = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix}$$

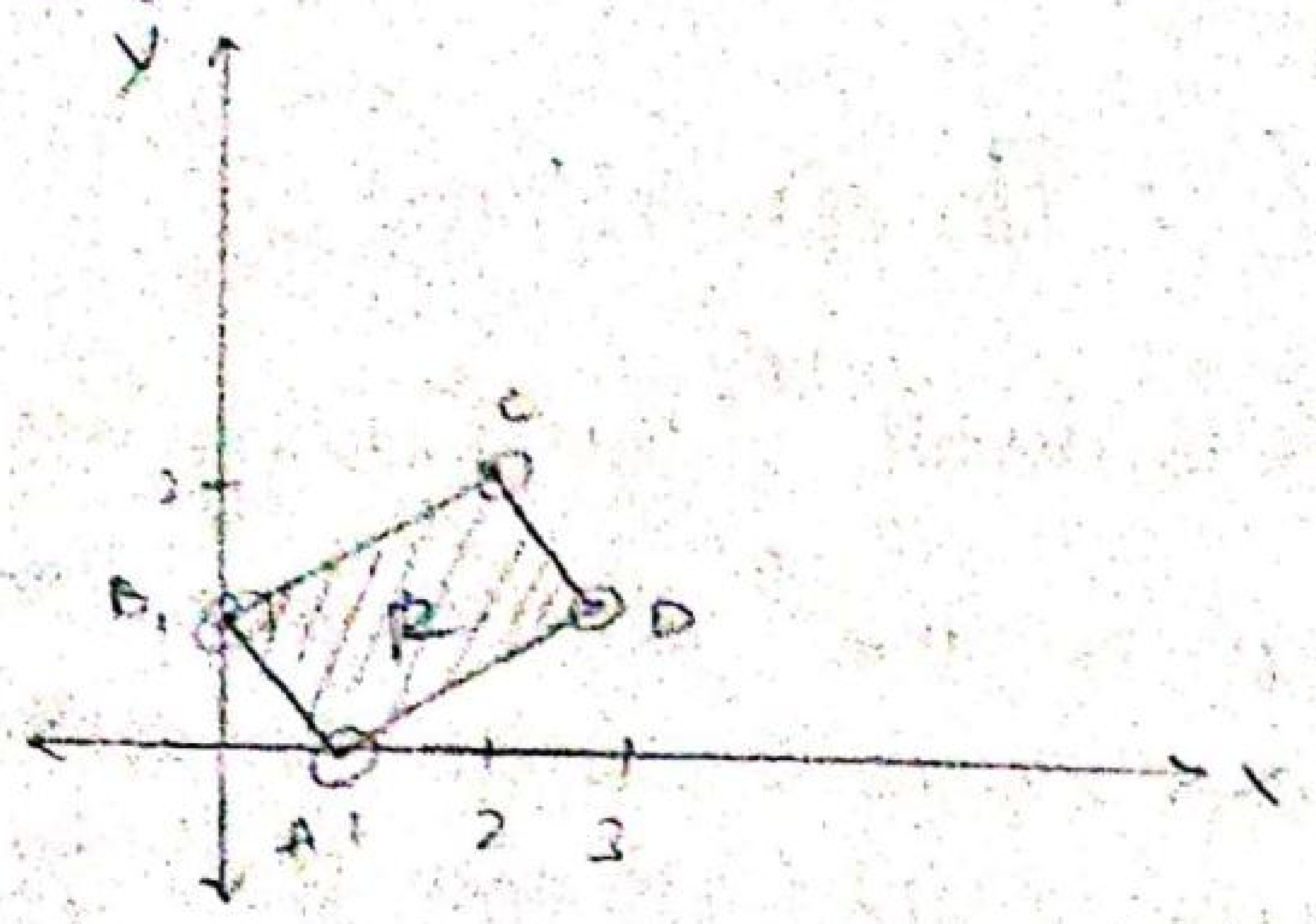
$$|J| = \left|-\frac{2}{9}-\frac{1}{9}\right|$$

$$|J| = \left|-\frac{1}{3}\right|$$

$$|J| = \frac{1}{3}$$

$\therefore dx\,dy = |J|\,du\,dv$

$dxdy = \frac{1}{3} du dv$



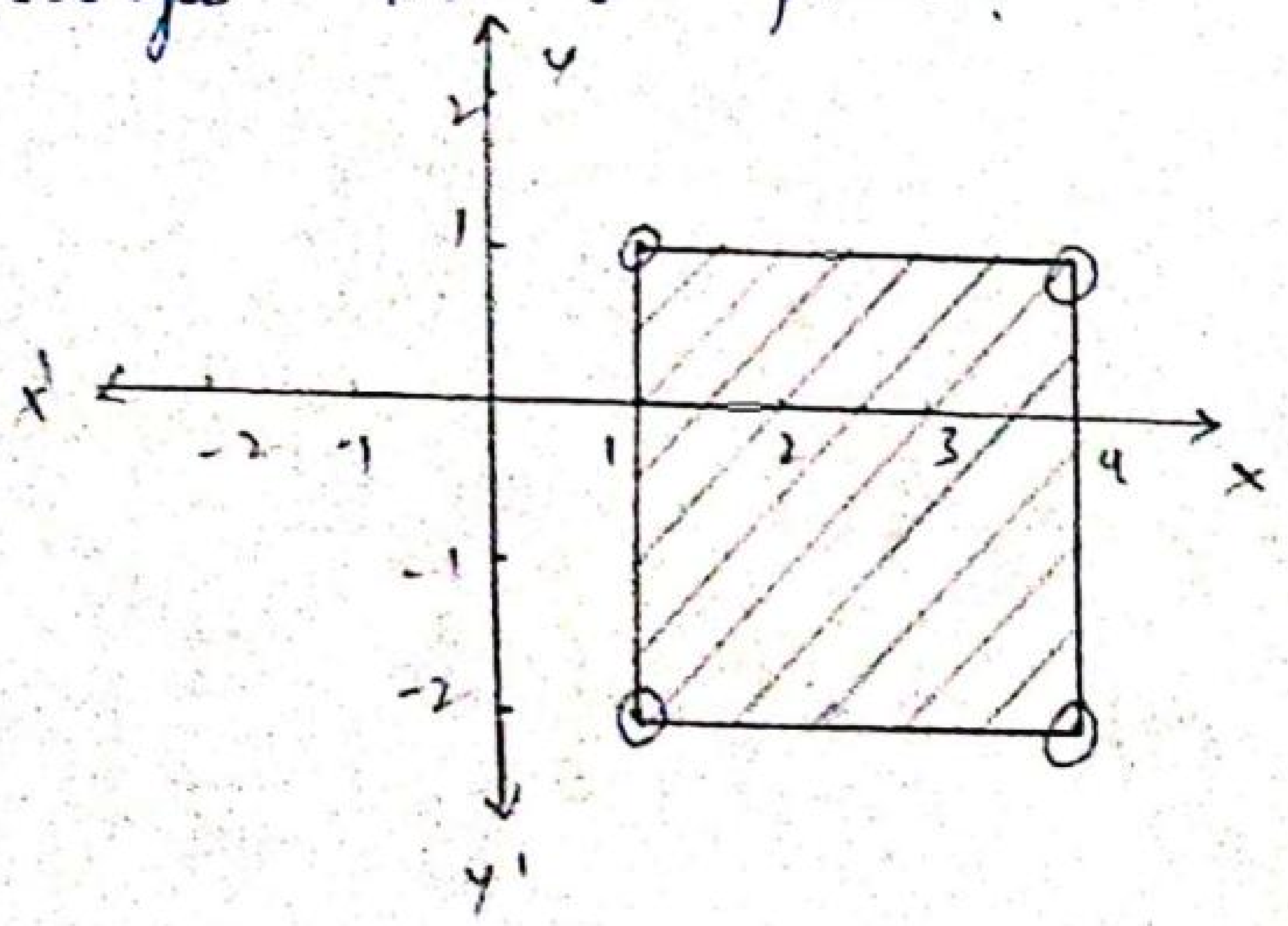
at $A(1,0)$, $u=1$, $v=1$ $\therefore A'(1,1)$

at $B(0,1)$, $u=1$, $v=-2$ $\therefore B'(1,-2)$

at $C(2,2)$, $u=4$, $v=-2$ $\therefore C'(4,-2)$

at $D(3,1)$, $u=4$, $v=1$ $\therefore D'(4,1)$

The region 'R' i.e., parallelogram ABCD in the XY-plane is transformed to R' where R' is rectangle in UV-plane.



$$\therefore \iint_R (x+y)^2 \, dxdy$$

$$= \int_{u=1}^{u=4}\int_{v=-2}^{v=1} u^2 \, |J| \, du dv$$

$$= \int_{u=1}^{4}\int_{-2}^{1} u^2 \cdot \frac{1}{3} \, du dv$$

$$= \frac{1}{3}\int_{u=1}^{4} u^2 du \cdot \int_{v=-2}^{1} dv$$

$$= \frac{1}{3}\left(\frac{u^3}{3}\right)_1^4 \left(\frac{v^{\cancel{2}}}{\cancel{2}}\right)_{-2}^{1}$$

$$(1+2)$$

$$= \frac{1}{3}\left(\frac{64}{3} - \frac{1}{3}\right)\cancel{\left(\frac{1}{2} - \frac{4}{2}\right)}$$

$$= \cancel{\frac{1}{3}\left(\frac{63}{3}\right)\left(\frac{-3}{2}\right)} (3)$$

$$= \cancel{\frac{-63}{6}}$$

$$= \frac{1}{3}\left(\frac{64}{3} - \frac{1}{3}\right)(3)$$

$$= \frac{63}{3}$$

3) Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} . dxdy$ by changing to polar coordinates. Hence show that

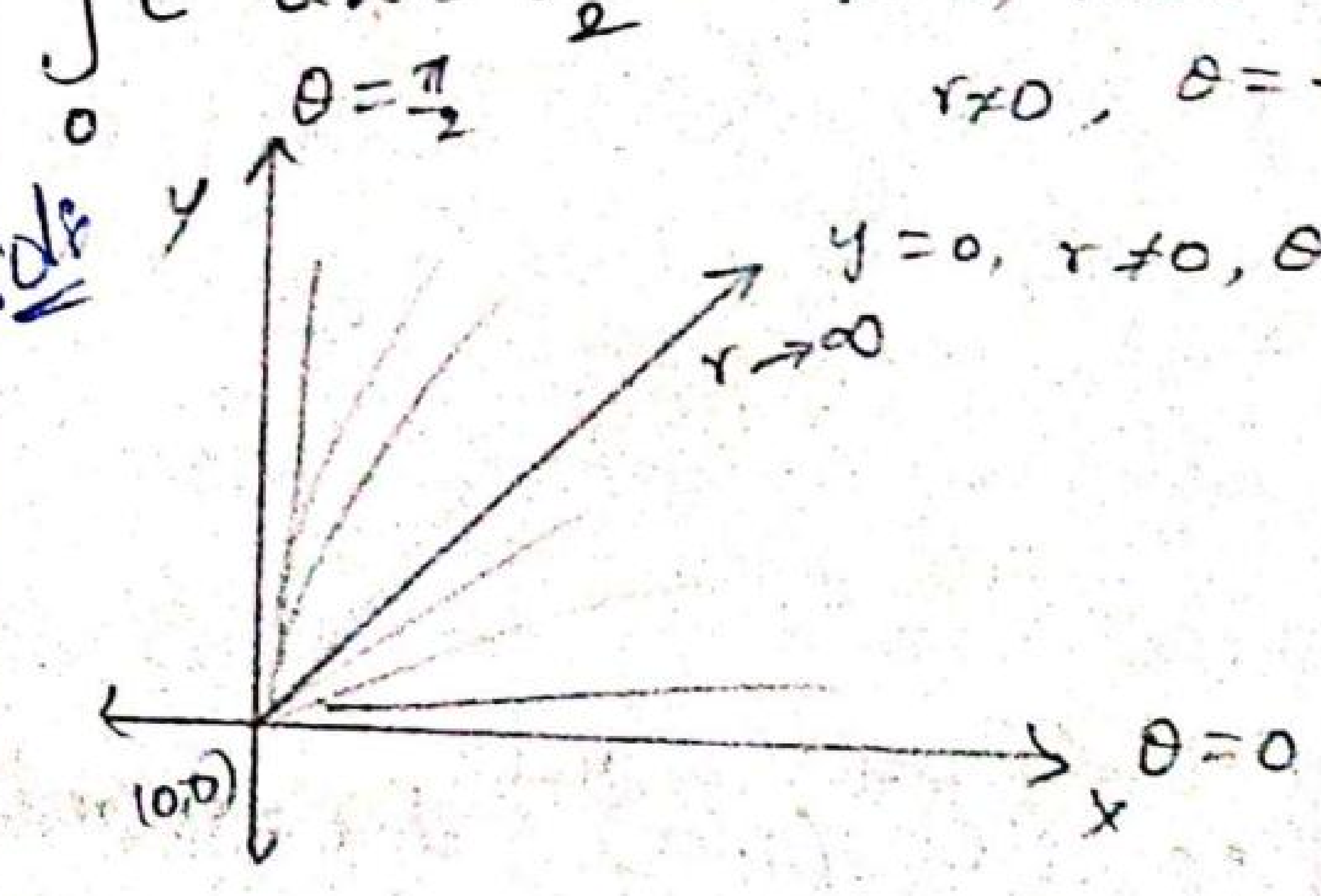
$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

$x=0$, $r\cos\theta = 0$

$r \neq 0$, $\theta = \frac{\pi}{2}$

$y=0$, $r\neq 0$, $\theta = 0$

sol:



changing to polar coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2+y^2=r^2$

also $dxdy = |J| \, dr d\theta$

$dxdy = r \, dr d\theta$

Since the region of integration in the entire 1st quadrant of xy-plane, 'r' varies from 0 to ∞ and 'θ' varies from 0 to $\frac{\pi}{2}$

$$\therefore \int_{\theta=0}^{\frac{\pi}{2}} \int_{0}^{\infty} e^{-(x^2+y^2)} dx\,dy$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{0}^{\infty} e^{-r^2} r\,dr\,d\theta.$$

$$= -\frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} \left[\int_{r=0}^{\infty} e^{-r^2}(-2r)dr\right] d\theta$$

$$= -\frac{1}{2}\int_{0}^{\frac{\pi}{2}} \left(e^{-r^2}\right)_0^{\infty} d\theta$$

$$= -\frac{1}{2}\int_{0}^{\frac{\pi}{2}} \left(e^{-\infty} - e^{-0}\right) d\theta$$

$$= -\frac{1}{2}\int_{0}^{\frac{\pi}{2}} (0-(+1))\, d\theta$$

$$= +\frac{1}{2}\int_{0}^{\frac{\pi}{2}} d\theta$$

$$= +\frac{1}{2}(\theta)_0^{\pi/2}$$

$$= +\frac{1}{2}\left(\frac{\pi}{2}-0\right)$$

$$= +\frac{\pi}{4}$$

$$\therefore \int_{0}^{\infty}\int_{0}^{\infty} e^{-(x^2+y^2)} dx\,dy = \frac{\pi}{4} \rightarrow (1)$$

Now, $\int_{0}^{\infty}\int_{0}^{\infty} e^{-(x^2+y^2)} dx\,dy.$

$$= \int_{0}^{\infty} e^{-x^2} dx \cdot \int_{0}^{\infty} e^{-y^2} dy$$

$$= \int_{0}^{\infty} e^{-x^2} dx \int_{0}^{\infty} e^{-x^2} dx$$

$$= \left(\int_{0}^{\infty} e^{-x^2} dx\right)^2 \rightarrow (2)$$

from (1) and (2)

$$\left(\int_{0}^{\infty} e^{-x^2} dx\right)^2 = \frac{\pi}{4}$$

$$\Rightarrow \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Hence proved

4) By changing into polar coordinates evaluate the following

(i) $\int_{0}^{2}\int_{0}^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{x^2+y^2}$

(ii) $\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2+y^2)\cdot \cancel{dy}\cdot dx\,dy.$

(i) Sol: Given that $y=0$ to $y=\sqrt{2x-x^2}$ and $x=0$ to $x=2$

i.e., $y=0$, $x^2+y^2=2x$, $x=0$ & $x=2$

$y=0 \Rightarrow r\sin\theta = 0$

$r\neq 0,\ \theta=0$

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$x=0 \Rightarrow r\cos\theta=0$

$r\neq 0,\ \theta=\frac{\pi}{2}$

$x^2+y^2=2x \Rightarrow r^2=2r\cos\theta$

$r=2\cos\theta$

Hence, in polar coordinates the given region is bounded by the curves $r=0, r=2\cos\theta$, $\theta=0, \theta=\frac{\pi}{2}$

$$\therefore \int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{x^2+y^2}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} \frac{\cancel{r}\cos\theta.\cancel{r}dr\,d\theta}{\cancel{r^2}}$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta\left(\int_0^{2\cos\theta} dr\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta\cdot(r)_0^{2\cos\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta(2\cos\theta-0)\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^2\theta\cdot d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}}\left(\theta+\frac{\sin 2\theta}{2}\right)_0^{\pi/2}$$

$$= \left[\frac{\pi}{2}+\frac{\sin(\frac{\pi}{2})}{2}\right]-\left(0+\overset{0}{\cancel{\sin\pi}}\right)$$

$$= \left(\frac{\pi}{2}+0\right)$$

$$= \frac{\pi}{2}$$

(ii) Given, that $y=0, y=1$,

$x=0, x=\sqrt{1-y^2}$

i.e., $y=0, y=1, x=0, x^2+y^2=1$

$x=0,\ r\cos\theta=0$

$r\neq0,\ \theta=\frac{\pi}{2}$

$y=0,\ r\sin\theta=0$

$r\neq0, \theta=0$

$x^2+y^2=1 \Rightarrow r^2=1$

$r=1$

Hence, in polar coordinates the given region is bounded by the curves $r=0, r=1$, $\theta=0, \theta=\frac{\pi}{2}$

$$\therefore \int_0^{1}\int_0^{\sqrt{1-y^2}}(x^2+y^2)\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r^2\,r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\int_0^1 r^3 dr\right]d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(\frac{r^4}{4}\right)_0^1 d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} d\theta$$

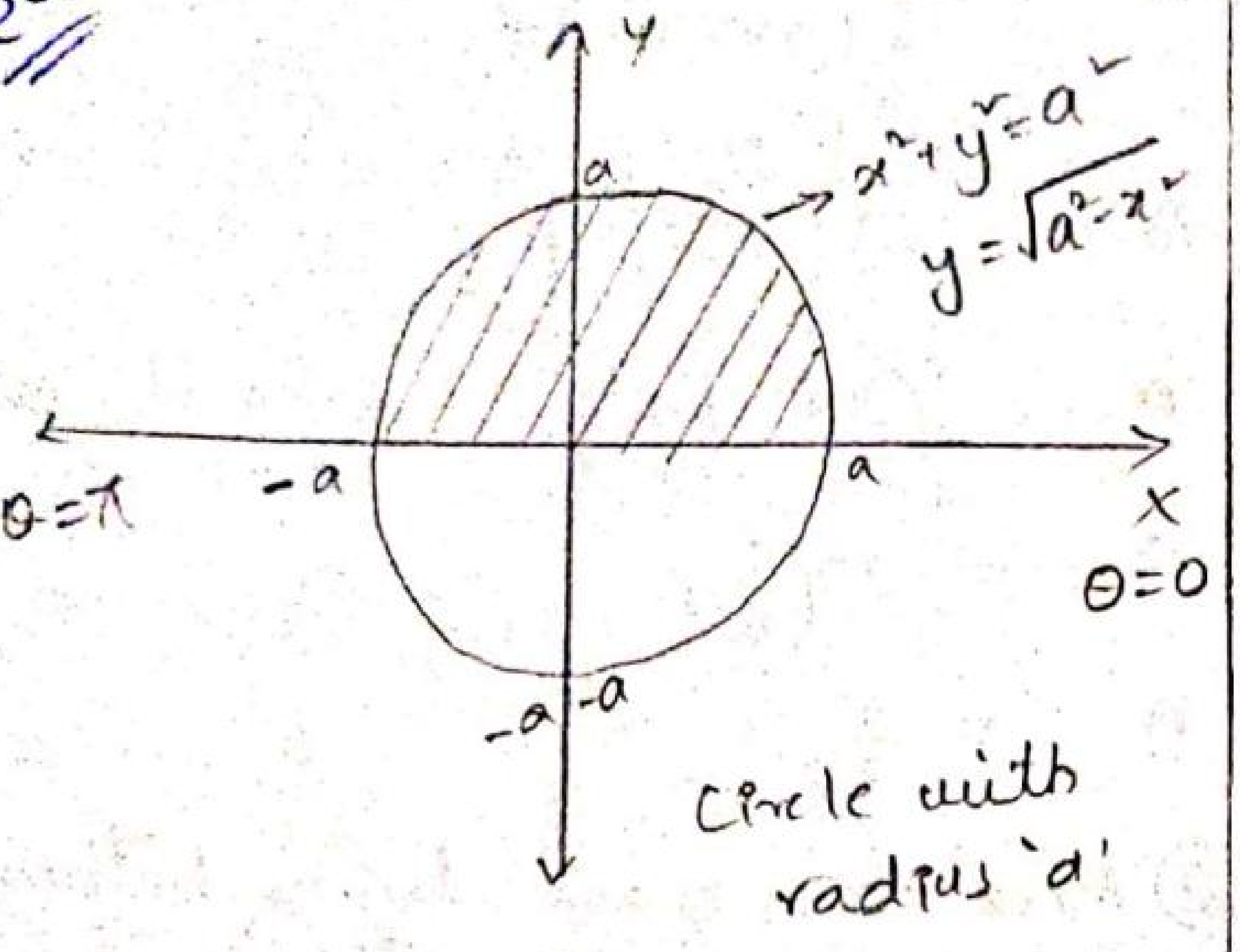
$= \frac{1}{4}(\theta)_0^{\pi/2}$

$= \frac{1}{4}\left(\frac{\pi}{2}\right)$

$= \frac{\pi}{8}$

5) Transform the following into cartesian form and hence evaluate $\int_0^{\pi}\int_0^{a} r^3 . \sin\theta . \cos\theta . dr d\theta$

Sol:



Given that $\theta=0, \theta=\pi$,

$r=0, r=a$

Here $r=a$, $x^2+y^2=r^2$

$x^2+y^2=a^2$

$\therefore r^2=a^2$

From $x^2+y^2=a^2$

$y=\sqrt{a^2-x^2}$

Here in cartesian coordinates the given region is bounded by the curves $x=-a$ to $x=a$ and $y=0$ to $y=\sqrt{a^2-x^2}$

$\therefore \int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta \, dr\, d\theta$

$= \int_0^{\pi}\int_0^{a} (r\sin\theta)(r\cos\theta)\, r\, dr\, d\theta$

$= \int_{x=-a}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}} xy\, dx\, dy$

$= \int_{x=-a}^{a}\left[\int_{y=0}^{\sqrt{a^2-x^2}} xy\, dy\right] dx$

$= \int_{x=-a}^{a} x\left[\int_{y=0}^{\sqrt{a^2-x^2}} y\, dy\right] dx$

$= \int_{x=-a}^{a} x.\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx.$

$= \int_{-a}^{a} x\left[\frac{a^2-x^2}{2} - 0\right] dx$

$= \frac{1}{2}\int_{-a}^{a}(a^2x - x^3)\, dx$

$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{-a}^{a}$

$= \frac{1}{2}\left[\left(\frac{a^2a^2}{2} - \frac{a^4}{4}\right) - \left(\frac{a^2a^2}{2} - \frac{a^4}{4}\right)\right]$

$= \frac{1}{2}\left(\cancel{\frac{a^4}{2}} - \cancel{\frac{a^4}{4}} - \cancel{\frac{a^4}{2}} + \cancel{\frac{a^4}{4}}\right)$

$= \frac{1}{2}(0)$

$= 0$

⇒ Area enclosed by plane curve using double integrals:

1) Area enclosed by plane curves in cartesian coordinates $A = \iint_R dx\,dy$

2) Area enclosed by plane curves in polar-coordinates is given by $A = \iint_R r\,dr\,d\theta$.

1) Find by double integration the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Sol: Given, $y = 4x - x^2 \rightarrow ①$ and
$y = x \rightarrow ②$

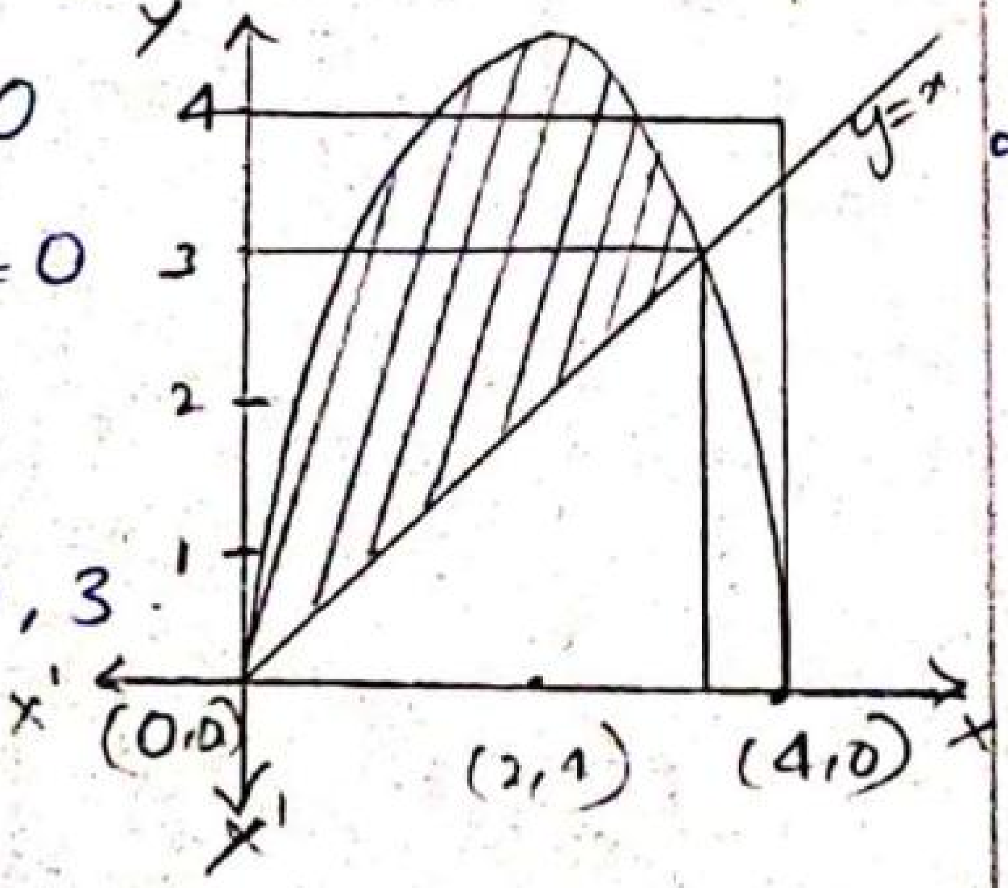
from ① and ② $x = 4x - x^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, 3$

from ② $y = 0, 3$.



∴ Intersection points $(0,0), (3,3)$

'x' varies from $x=0$ to $x=3$

'y' varies from $y=0$ to $y=4x-x^2$.

Required area $= A = \iint_R dx\,dy$

$$A = \int_{x=0}^{3}\int_{y=x}^{y=4x-x^2} dy\,dx$$

$$A = \int_{x=0}^{3}\left[\int_{y=x}^{y=4x-x^2} dy\right]dx.$$

$$A = \int_0^3 [y]_x^{4x-x^2}\,dx$$

$$A = \int_0^3 (4x - x^2 - x)\,dx$$

$$A = \int_0^3 (3x - x^2)\,dx$$

$$A = \frac{3x^2}{2} - \frac{x^3}{3}\Big|_0^3$$

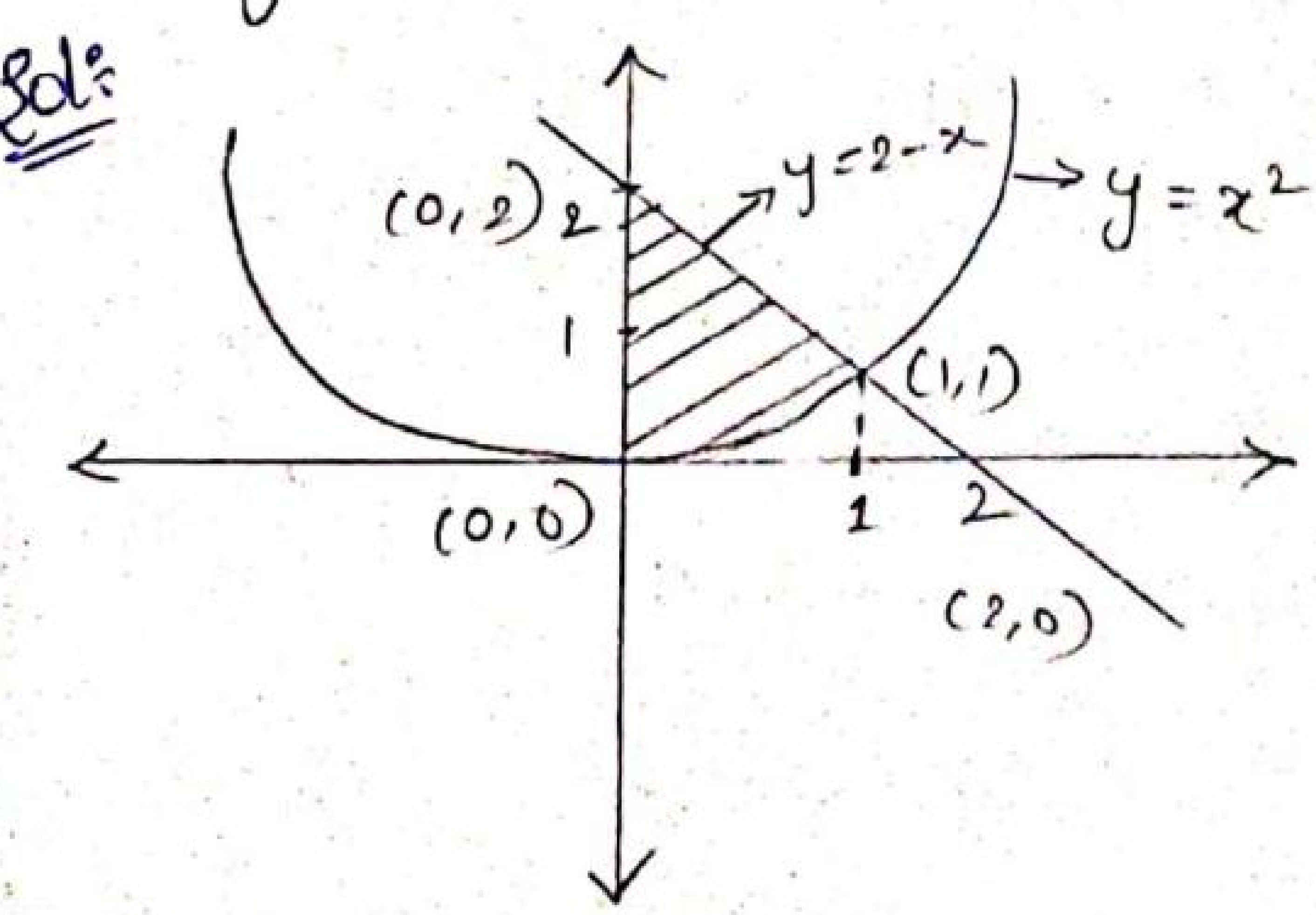
$$A = \left(\frac{27}{2} - \frac{27}{3}\right)$$

$$A = 27\left(\frac{1}{6}\right)$$

$$A = \frac{9}{2}$$

2) Find the area lying between $y = x^2$ and the line $x + y + 2 = 0$ and y-axis and x-axis.

Sol:



Given $y = x^2$, $x + y - 2 = 0$ and y-axis.

Intersection points are $(1,1)$ and $(0,2)$

x- varies from $x=0$ to $x=1$

y- varies from $y = x^2$ to $y = 2-x$

$\therefore$ Required area $= A = \iint_R dx\,dy$

$$A = \int_{x=0}^{1}\int_{y=x^2}^{y=2-x} dy\,dx.$$

$$A = \int_{x=0}^{1} [y]_{x^2}^{2-x}\, dx.$$

$$A = \int_0^1 (2-x-x^2)\,dx.$$

$$A = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

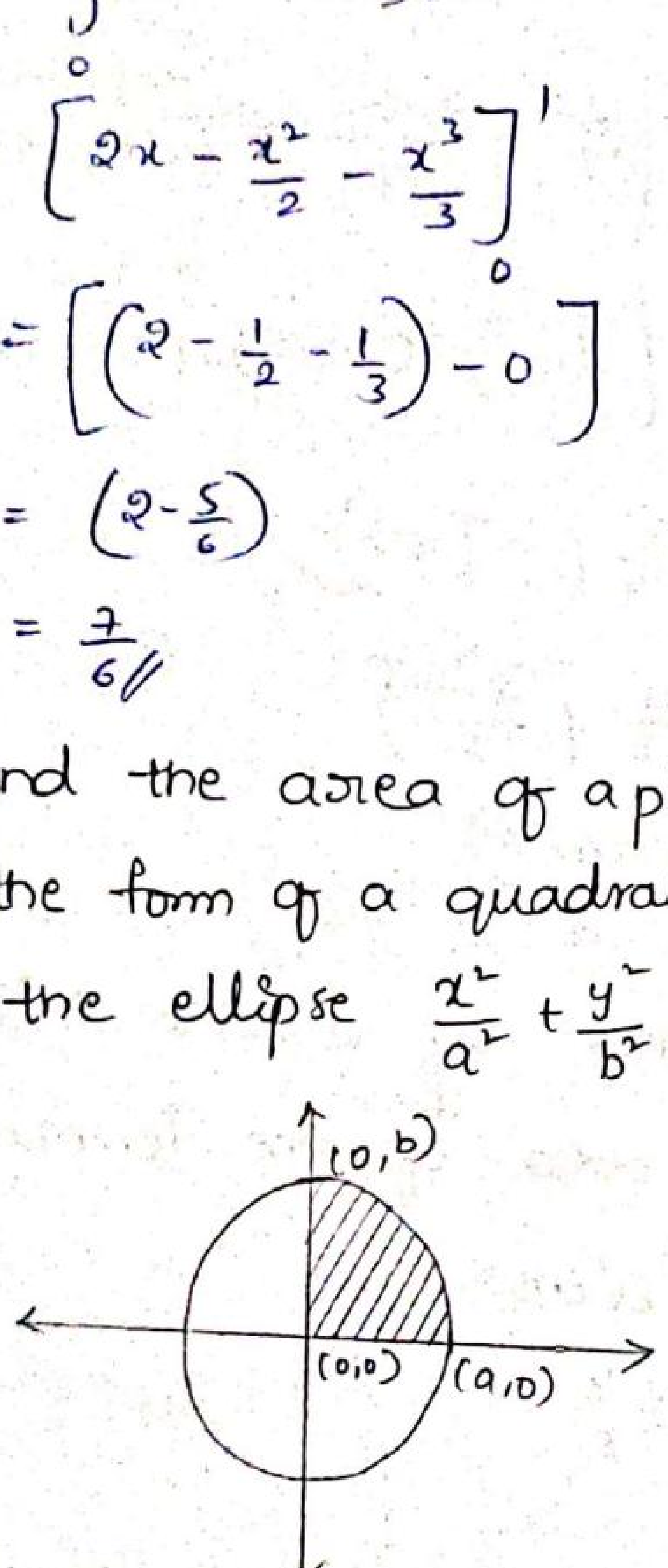
$$A = \left[\left(2 - \frac{1}{2} - \frac{1}{3}\right) - 0\right]$$

$$A = \left(2 - \frac{5}{6}\right)$$

$$A = \frac{7}{6}$$

3) Find the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol:



Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

'x' varies from $x=0$ to $x=a$.

'y' varies from $y=0$ to $y = \frac{b}{a}\sqrt{a^2-x^2}$.

$\therefore$ Required area $= \iint_R dx\,dy$

$$= \int_{x=0}^{a}\int_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$$

$$= \int_{x=0}^{a} [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\, dx.$$

$$= \int_{x=0}^{a} \frac{b}{a}\sqrt{a^2-x^2}\, dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

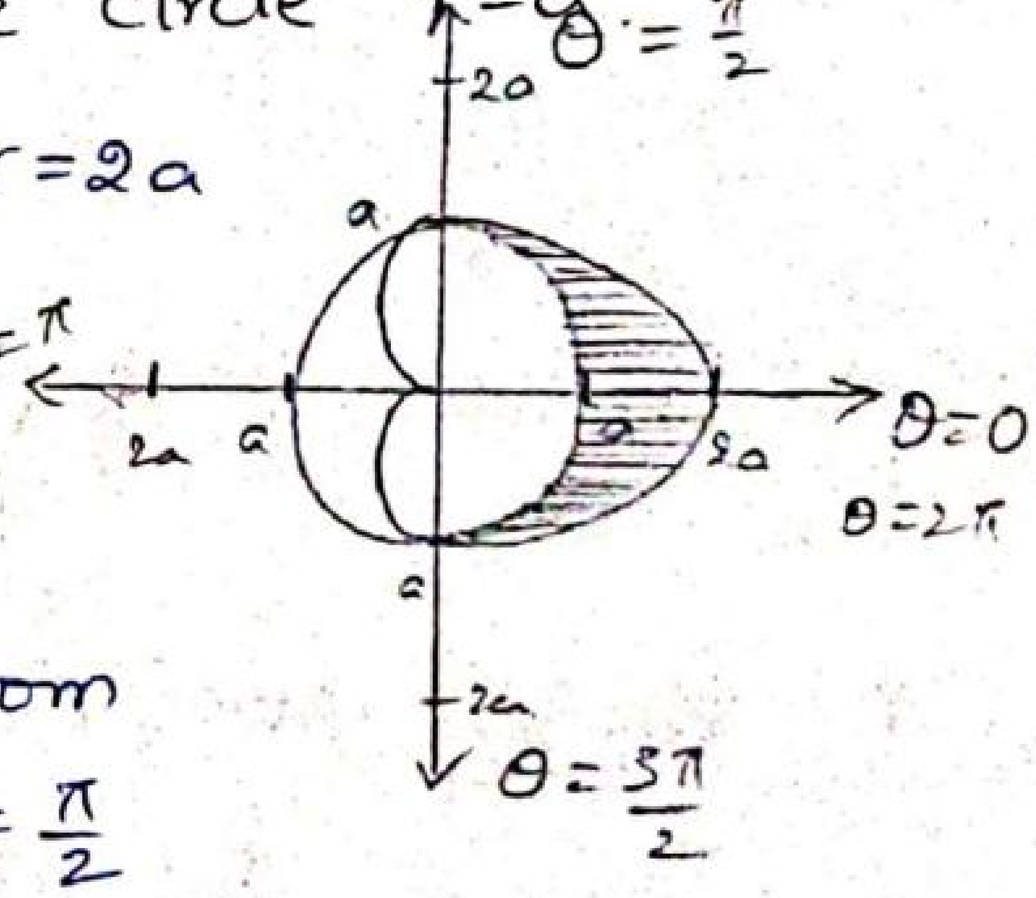
$$= \frac{b}{a}\left[\frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right) - (0+0)\right]$$

$$= \frac{b}{a}\left(0 + \frac{a^2}{2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{b}{a}\left(\frac{\pi a^2}{4}\right)$$

$$= \frac{\pi ab}{4} \text{ units}$$

4) Find the area lying inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.



Sol: $\theta = 0,\ r = 2a$

$\theta = \frac{\pi}{2},\ r = a$

$\theta = \pi,\ r = 0$

'θ' varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$

'r' varies from $r = a$ to $r = a(1+\cos\theta)$

$\therefore$ required area $= A = \iint_R r\,dr\,d\theta.$

$$A = 2\int_0^{\pi/2}\int_a^{r=a(1+\cos\theta)} r\,dr\,d\theta.$$

$$A = 2\int_0^{\pi/2}\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta.$$

$$A = \frac{\cancel{2}}{\cancel{2}}\int_0^{\pi/2}\left[a^2(1+\cos\theta)^2 - a^2\right] d\theta.$$

$$= a^2 \int_0^{\pi/2} (\cancel{1}+\cos^2\theta+2\cos\theta-\cancel{1})\, d\theta$$

$$= a^2 \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2} + 2\cos\theta\right) d\theta$$

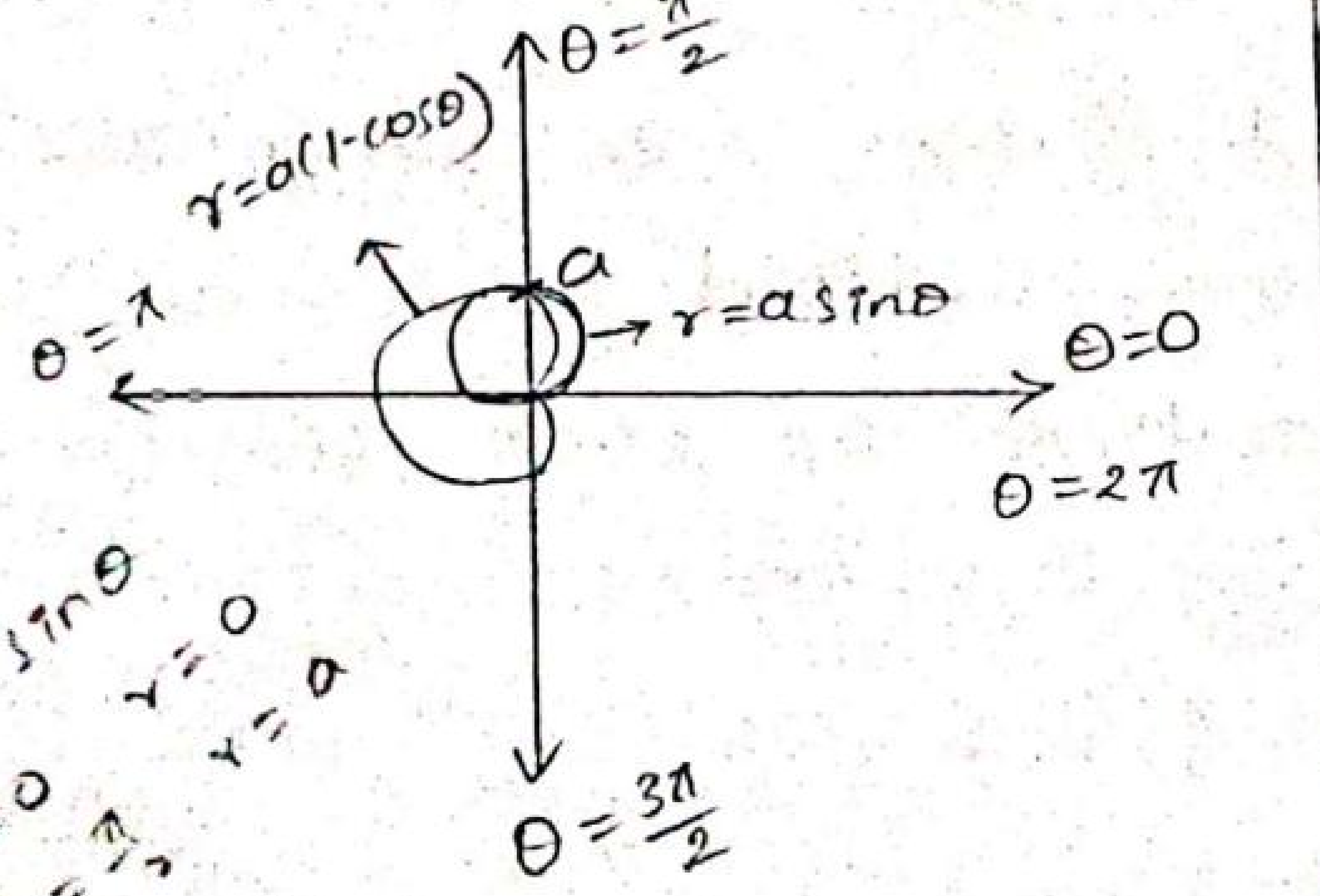
$$= a^2 \left(\frac{\pi}{4} + \frac{1}{2}\theta + \frac{1}{2}\,\frac{\sin 2\theta}{2} + 2\sin\theta\right)_0^{\pi/2}$$

$$= a^2 \left[\frac{\pi}{4} + \frac{1}{2}\left(\frac{0}{2}\right) + 2\sin\left(\frac{\pi}{2}\right)\right] - (0)$$

$$= a^2 \left(\frac{\pi}{4} + 2\right)$$

$$= \frac{a^2}{4}(\pi+8) \text{ units}$$

5) Find by double integration the area lying inside the Circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$



$r = a\sin\theta$, $r=0$
$\theta = 0$, $r = 0$
$\theta = \frac{\pi}{2}$

'r' varies from $r = a(1-\cos\theta)$ to $r = a\sin\theta$

'θ' varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$

~~Reqesed~~ Required area

$$A = \int_{\theta=0}^{\pi/2} \int_{r=a(1-\cos\theta)}^{r=a\sin\theta} r\, dr\, d\theta$$

$$A = \int_{\theta=0}^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} \left[a^2\sin^2\theta - a^2(1-\cos\theta)^2\right] d\theta$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi/2} (\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta)\, d\theta$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi/2} \left(\frac{1-\cos 2\theta}{2} - 1 - \left(\frac{1+\cos 2\theta}{2}\right) + 2\cos\theta\right) d\theta$$

$$= \frac{a^2}{2}\left[\cancel{\frac{\theta}{2}} - \frac{\sin 2\theta}{4} - \theta - \cancel{\frac{\theta}{2}} - \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\left(0 - \frac{\pi}{2} - 0 + 2(1)\right) - (0-0-0+0)\right]$$

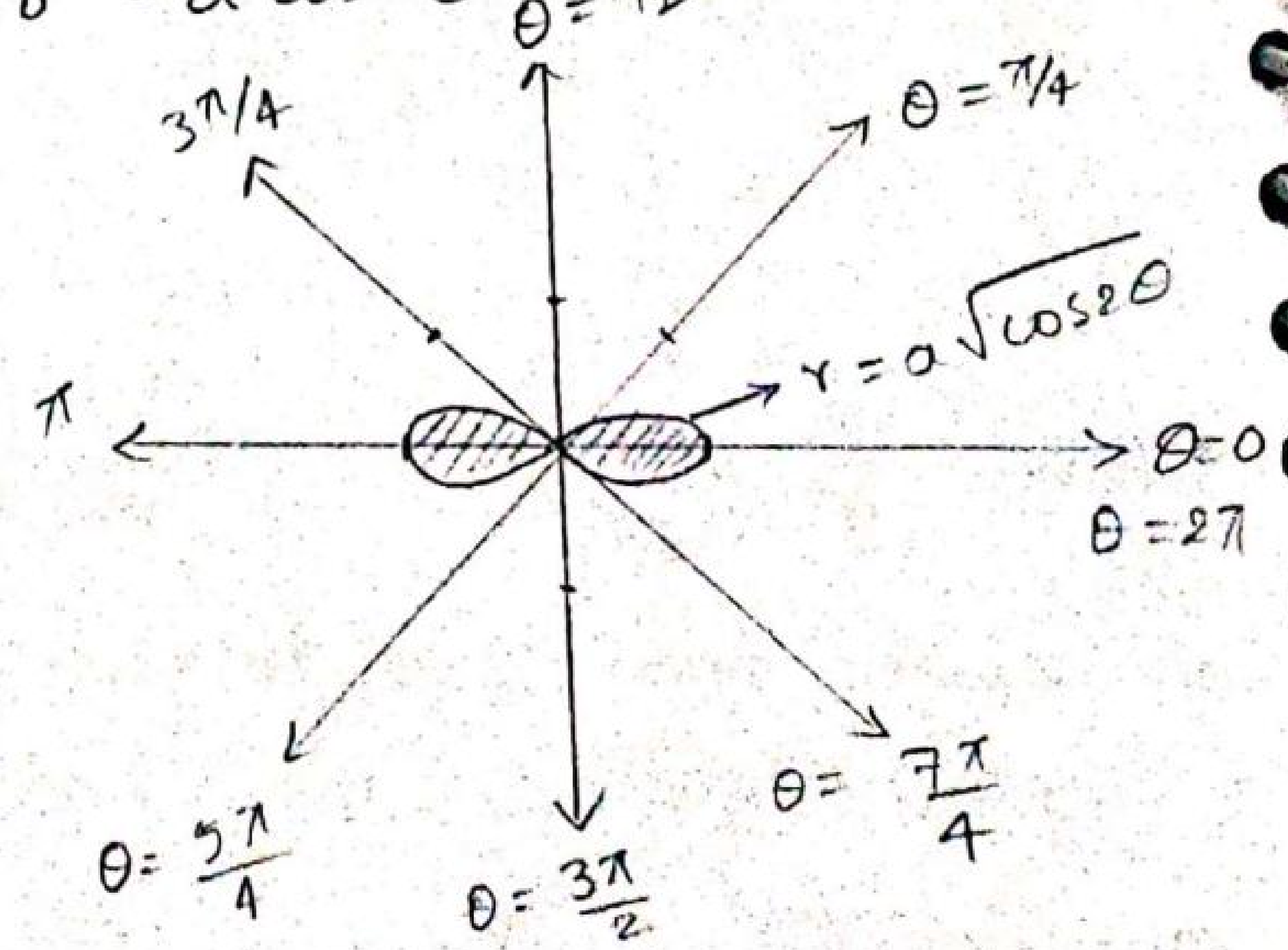
$$= \frac{a^2}{2}\left(-\frac{\pi}{2} + 2\right)$$

$$= \frac{a^2}{2}\left(\frac{4-\pi}{2}\right)$$

$$= \frac{a^2(4-\pi)}{4}$$

6) Find by double integration the area of the lemniscate

$$r^2 = a^2\cos 2\theta$$



$\theta = 0$, $r = a$

$\theta =$

'θ' var
'r' var
$\therefore$ Rec
A
A
A
A
A

'θ' varies from $\theta = 0$ to $\theta = \frac{\pi}{4}$

'r' varies from $r = 0$ to $r = a\sqrt{\cos 2\theta}$

$\therefore$ Required area

$$A = 4 \int_{0}^{\pi/4} \int_{r=0}^{r=a\sqrt{\cos 2\theta}} r\,dr\,d\theta.$$

$$A = 4 \int_{\theta=0}^{\pi/4} \left(\frac{r^2}{2}\right)_0^{a\sqrt{\cos 2\theta}} d\theta.$$

$$A = 2 \int_{0}^{\pi/4} a^2 \cos 2\theta . d\theta$$

$$A = 2a^2 \int_{\theta=0}^{\pi/4} \cos 2\theta\, d\theta$$

$$A = 2a^2 \left(\frac{\sin 2\theta}{2}\right)_0^{\pi/4}$$

$$A = a^2 \left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$A = a^2(1)$$

$$A = a^2 \text{ units}$$

TRIPLE INTEGRALS

Evaluate $\int_{x=x_1}^{x_2} \int_{y=y_1}^{y_2} \int_{z=z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$

If x_1, x_2 are constants ; y_1, y_2 are either constants or functions of 'x' ; z_1, z_2 are either constants or functions of x, y, then integrate $f(x,y,z)$ first w.r.t. 'z' keeping x, y as constants the resulting expression w.r.t. 'y' and then the resulting expression w.r.t. 'x'. Therefore.

$$I = \int_{x_1}^{x_2} \int_{y=y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

(I) Evaluate the following.

1) $\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz.$

Sol: Given $\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz$

$$= \int_0^a \int_0^b \left[\frac{x^3}{3} + y^2x + z^2x\right]_0^c dy\,dz.$$

$$= \int_0^a \int_0^b \left[\frac{c^3}{3} + y^2c + z^2c\right] dy\,dz.$$

$$= \int_0^a \left(\frac{c^3}{3}.y + \frac{y^3}{3}c + z^2c.y\right)_0^b dz$$

$$= \int_0^a \left(\frac{c^3}{3} b + \frac{b^3}{3} \cdot c + z^2 \cdot c \cdot b\right) dz$$

$$= \frac{c^3}{3} \cdot bz + \frac{b^3 c}{3} \cdot z + \frac{z^3}{3} \cdot bc \Big|_0^a$$

$$= \left[\frac{abc^3}{3} + \frac{ab^3c}{3} + \frac{a^3bc}{3}\right]$$

$$= \frac{abc}{3}(a^2+b^2+c^2)$$

2) $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \cdot dz\, dx\, dy$

Sol: Given, $\int_0^1 \int_{x=y^2}^1 \int_{z=0}^{z=1-x} x \cdot dz \cdot dx \cdot dy$

$$= \int_0^1 \int_{x=y^2}^1 [xz]_0^{1-x}\, dx dy$$

$$= \int_0^1 \int_{x=y^2}^1 x(1-x)\, dx dy$$

$$= \int_0^1 \int_{x=y^2}^1 (x-x^2)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y^2}^1 dy$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right) dy$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy$$

$$= \left[\frac{y}{6} - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1$$

$$= \left(\frac{1}{6} - \frac{1}{10} + \frac{1}{21}\right) - 0$$

$$= \frac{4}{35}$$

3) $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$

Sol: Given, $\int_{z=-1}^1 \int_{x=0}^{x=z} \int_{y=x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$

$$= \int_{z=-1}^1 \int_{x=0}^{x=z} \left(xy + \frac{y^2}{2} + zy\right)_{x-z}^{x+z} dx \cdot dz$$

$$= \int_{z=-1}^1 \int_{x=0}^{z} \left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z)\right] - \left[x(x-z) + \frac{(x-z)^2}{2} + z(x-z)\right] dx\, dz$$

$$= \int_{z=-1}^1 \int_{x=0}^{z} \left[\cancel{x^2} + xz + \cancel{\frac{x^2}{2}} + \cancel{\frac{z^2}{2}} + xz + zx + z^2 - \cancel{x^2} + xz - \cancel{\frac{x^2}{2}} - \cancel{\frac{z^2}{2}} + x\cancel{z} - x\cancel{z} + z^2\right] dx\, dz$$

$$= \int_{z=-1}^1 \int_{x=0}^{z} (2z^2 + 4xz)\, dx\, dz$$

$$= \int_{z=-1}^1 \left(2z^2 \cdot x + \frac{4zx^2}{2}\right)_0^z dz$$

$$= \int_{z=-1}^1 \left(2z^2 \cdot z + \frac{4z^3}{2}\right) dz$$

$$= \int_{z=-1}^1 (2z^3$$

$$= \left[\frac{z^3}{3}\right.$$

$$= \int_{z=-1}^1 4$$

$$= \left[4\right.$$

$$= 4$$

$$= 0$$

4) $\int_0$

Sol:

$$= \int_{x=0}$$

$$= \int_{x=0}$$

$$=$$

$$=$$

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$$= \int_{z=-1}^{1} (2z^3 + 2z^3)\, dz$$

$$= \left[\frac{z^4}{2} + \frac{2z^4}{4}\right]_{-1}^{1}$$

$$= \int_{z=-1}^{1} 4z^3\, dz$$

$$= \left[4\frac{z^4}{4}\right]_{-1}^{1}$$

$$= 4\left[\frac{1}{4} - \frac{1}{4}\right]$$

$$= 0$$

4) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dx\, dy\, dz$

Sol: Given, $\int_{x=0}^{1} \int_{y=0}^{y=\sqrt{1-x^2}} \int_{z=0}^{z=\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left(xy\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy\, dx.$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy\frac{(1-x^2-y^2)}{2}\, dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} (xy - x^3y - xy^3)\, dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \left(x\frac{y^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right)_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{4}\int_{x=0}^{1} \left(xy^2 - x^3y^2 - \frac{xy^4}{2}\right)_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{4}\int_{x=0}^{1} \left(x(1-x^2) - x^3(1-x^2) - \frac{x(1-x^2)^2}{2}\right) dx$$

$$= \frac{1}{4}\int_0^1 \left(x - x^3 - x^3 + x^5 - \frac{x(1+x^4-2x^2)}{2}\right) dx$$

$$= \frac{1}{4}\int_0^1 \left(x - 2x^3 + x^5 - \frac{(x + x^5 - 2x^3)}{2}\right) dx$$

$$= \frac{1}{8}\int_0^1 \left(2x - 4x^3 + 2x^5 - x - x^5 + 2x^3\right) dx$$

$$= \frac{1}{8}\int_0^1 \left(x - 2x^3 + x^5\right) dx$$

$$= \frac{1}{8}\left(\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6}\right)_0^1$$

$$= \frac{1}{8}\left(\frac{1}{2} - \frac{2}{4} + \frac{1}{6}\right)$$

$$= \frac{1}{16}\left(1 - \frac{\cancel{2}}{\cancel{2}} + \frac{1}{3}\right)$$

$$= \frac{1}{48}$$

5) $\int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz$

Sol: Given, $\int_{x=0}^{\log 2} \int_{y=0}^{x} \int_{z=0}^{x+\log y} e^{x+y+z}\, dz\, dy\, dx$

$$= \int_{x=0}^{\log 2} \int_{y=0}^{x} \left[e^{x+y}(e^z)\right]_0^{x+\log y} dy\, dx$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x} e^{x+y}\left(e^{x+\log y} - e^{x}\right) dy\,dx$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x} e^{x+y}\left(y.e^{x} - 1\right) dy\,dx.$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x} \left(y.e^{2x+y} - e^{x+y}\right) dy\,dx$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x} \left(y.e^{2x}.e^{y} - e^{x}.e^{y}\right) dy\,dx$$

$$\left(\because \int ye^{y} = y.e^{y} - 1\,e^{y} = (y-1)e^{y}\right)$$

$$= \int_{x=0}^{\log 2} \left[e^{2x}(y-1)e^{y} - e^{x}.e^{y}.\right]_{0}^{x} dx$$

$$\int_{x=0}^{\log 2} \left(e^{2x}(x-1)e^{x} - e^{x}.e^{x}\right) - \left(e^{2x}(0-1)e^{0} - e^{x}.e^{0}\right) dx$$

$$= \int_{x=0}^{\log 2} \left(xe^{3x} - e^{3x} - \cancel{e^{2x}} + \cancel{e^{2x}} + e^{x}\right) dx$$

$$= \int_{x=0}^{\log 2} \left[(x-1)e^{3x} + e^{x}\right].dx$$

$$\left(\because (e^{3x})(x-1) = (x-1)\frac{e^{3x}}{3} - 1.\frac{e^{3x}}{9}\right)$$

$$= \left[(x-1)\frac{e^{3x}}{3} - \frac{e^{3x}}{9} + e^{x}\right]_{0}^{\log 2}$$

$$= \left[(\log 2 - 1)\frac{e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} + e^{\log 2}\right] - \left[(0-1)\frac{e^{0}}{3} + \frac{e^{0}}{9} + e^{0}\right]$$

$$= \left[(\log 2 - 1)\frac{8}{3} - \frac{8}{9} + 2\right] - \left(-\frac{1}{3} - \frac{1}{9} + 1\right)$$

$$= \frac{8}{3}\log 2 - \frac{8}{3} - \frac{8}{9} + 2 + \frac{1}{3} + \frac{1}{9} - 1$$

$$= \frac{8}{3}\log 2 - \frac{19}{9}$$

(6) $\int_{y=1}^{e}\int_{x=1}^{\log y}\int_{z=1}^{e^{x}} \log z.\,dz.\,dx.dy.$

Sol: $\int_{y=1}^{e}\int_{x=1}^{\log y}\left[\int_{z=1}^{e^{x}} \log z.\,dz\right] dx\,dy$

$$= \int_{y=1}^{e}\int_{x=1}^{\log y} \left(z\log x - z\right)_{1}^{e^{x}} dx\,dy$$

$$\left(\because \int \log x\,dx = x(\log x - 1)\right)$$

$$= \int_{y=1}^{e}\int_{x=1}^{\log y} \left[\left(e^{x}\log_{e} e^{x} - e^{x}\right) - (0-1)\right] dx.dy$$

$$= \int_{y=1}^{e}\int_{x=1}^{\log y} \left(xe^{x} - e^{x} + 1\right) dx\,dy$$

$\Big]_0^{\log 2}$

$+ e^{\log 2}\Big]$

$\Big]$

$+\frac{1}{9}-1$

$$= \int_{y=1}^{e} \left[xe^x - e^x - e^x + x\right]_1^{\log y} dy$$

$$\left(\because \int xe^x dx = xe^x - 1.e^x = xe^x - e^x\right)$$

$$= \int_{y=1}^{e} \left(xe^x - 2e^x + x\right)_1^{\log y} dy$$

$$= \int_{y=1}^{e} \left[\left(\log y . e^{\log y} - 2e^{\log_e y} + \log y\right) - (e - 2e + 1)\right] dy$$

$$= \int_{y=1}^{e} \left[y\log y - 2y + \log y + e - 1\right] dy$$

$$= \left[\log y \int y dy - \int\left[\frac{1}{y}\int y dy\right]dy - y^2 + y\log y - y + ey - y\right]_1^e$$

$$= \left[\frac{y^2}{2}\log y - \frac{y^2}{4} - y^2 + y\log y - 2y + ey\right]_1^e$$

$$= \left[\frac{y^2}{2}\log y - \frac{5y^2}{4} + y\log y + (e-2)y\right]_1^e$$

$$= \left[\frac{e^2}{2}\log_e e - \frac{5e^2}{4} + e\log e + (e-2)e\right] - \left[0 - \frac{5}{4} + 0 + (e-2)1\right]$$

$$= \frac{e^2}{2} - \frac{5e^2}{4} + \cancel{e} + e^2 - 2e + \frac{5}{4} - \cancel{e} + 2$$

$$= e^2\left(\frac{1}{2} - \frac{5}{4} + 1\right) - 2e + \frac{5}{4} + 2$$

$$= e^2\left(\frac{1}{4}\right) - 2e + \frac{13}{4}$$

$$= \frac{e^2 - 8e + 13}{4}$$

7) $\int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{2}} r\,dz\,dr\,d\theta.$

Soln $= \int_{\theta=0}^{\pi/2} \int_{r=0}^{r=a\sin\theta} \int_{z=0}^{z=\frac{a^2-r^2}{2}} r\,dz\,dr\,d\theta.$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{r=a\sin\theta} \left[\int_{z=0}^{z=\frac{a^2-r^2}{2}} r\,dz\right] dr\,d\theta.$$

$$\cancel{= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{r\sin\theta} \left(\frac{ar^2 - r^3}{2}\right) dr\,d\theta.}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{r\sin\theta} r\left(\frac{a^2-r^2}{2}\right) dr\,d\theta.$$

$$= \int_0^{\frac{\pi}{2}} \int_{r=0}^{a\sin\theta} \left(\frac{a^2 r - r^3}{2}\right) dr\,d\theta.$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{a^2 r^2}{2} - \frac{r^4}{4}\right)_0^{a\sin\theta} d\theta.$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\left(\frac{a^2 a^2 \sin^2\theta}{2} - \frac{a^4 \sin^4\theta}{4}\right)\right] d\theta$$

$$= \frac{a^4}{4}\left[\int_0^{\pi/2} \sin^2\theta\,d\theta - \frac{1}{2}\int_0^{\pi/2} \sin^4\theta . d\theta\right]$$

$= \frac{a^4}{4}\left(\int_0^{\pi/2} \frac{(1-\cos 2\theta)}{2} d\theta - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$

$= \frac{a^4}{4}\left[\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/2} - \frac{3\pi}{32}\right]$

$= \frac{a^4}{4}\left[\frac{\pi}{4} - \frac{3\pi}{32}\right]$

$= \frac{a^4}{4}\left(\frac{8\pi - 3\pi}{32}\right)$

$= \frac{a^4}{4}\left(\frac{5\pi}{32}\right)$

$= \frac{5a^4\pi}{128}$ ~~units~~

8) $\int_{z=0}^{2}\int_{y=1}^{z}\int_{x=0}^{yz} xyz \, dz\, dy\, dx.$

Sol:

Volume of solids by Triple Integrals:

The volume of solid in

1) Cartesian coordinates is

$\iiint_V dx\, dy\, dz.$

2) Cylindrical coordinates is

$\iiint_V r . dp . d\phi . dz$

3) Spherical coordinates

$\iiint_V r^2 \sin\theta \, dr\, d\theta\, d\phi$

8) Sol: Given, $\int_{z=0}^{2}\int_{y=1}^{z}\int_{x=0}^{yz} xyz \, dx\, dy\, dz$

$= \int_{z=0}^{2}\int_{y=1}^{z}\left[yz\frac{x^2}{2}\right]_0^{yz} dy\, dz$

$= \int_{z=0}^{2}\int_{y=1}^{z} \frac{yz\, y^2 z^2}{2} dy\, dz$

$= \frac{1}{2}\int_{z=0}^{2}\int_{y=1}^{z} y^3 z^3 \, dy\, dz$

$= \frac{1}{2}\int_{z=0}^{2}\left[\int_{y=1}^{z} y^3 z^3 \, dy\right] dz.$

$= \frac{1}{2}\int_{z=0}^{2}\left(\frac{y^4}{4} z^3\right]_{y=1}^{z} dz.$

$= \frac{1}{2}\int_{z=0}^{2}\left(\frac{z^4}{4}\cdot z^3 - \frac{1}{4}z^3\right] dz.$

$= \frac{1}{2}\cdot\frac{1}{4}\int_{z=0}^{2}\left(z^7 - z^3\right) dz$

$= \frac{1}{8}\int_{z=0}^{2}\left(z^7 - z^3\right) dz$

$= \frac{1}{8}\left(\frac{z^8}{8} - \frac{z^4}{4}\right)_0^2$

$= \frac{1}{8}\left(\frac{2^8}{8} - \frac{2^4}{4}\right)$

xdydz

$$= \frac{2^4}{8\cdot 4}\left(\frac{2^4}{2} - 1\right)$$

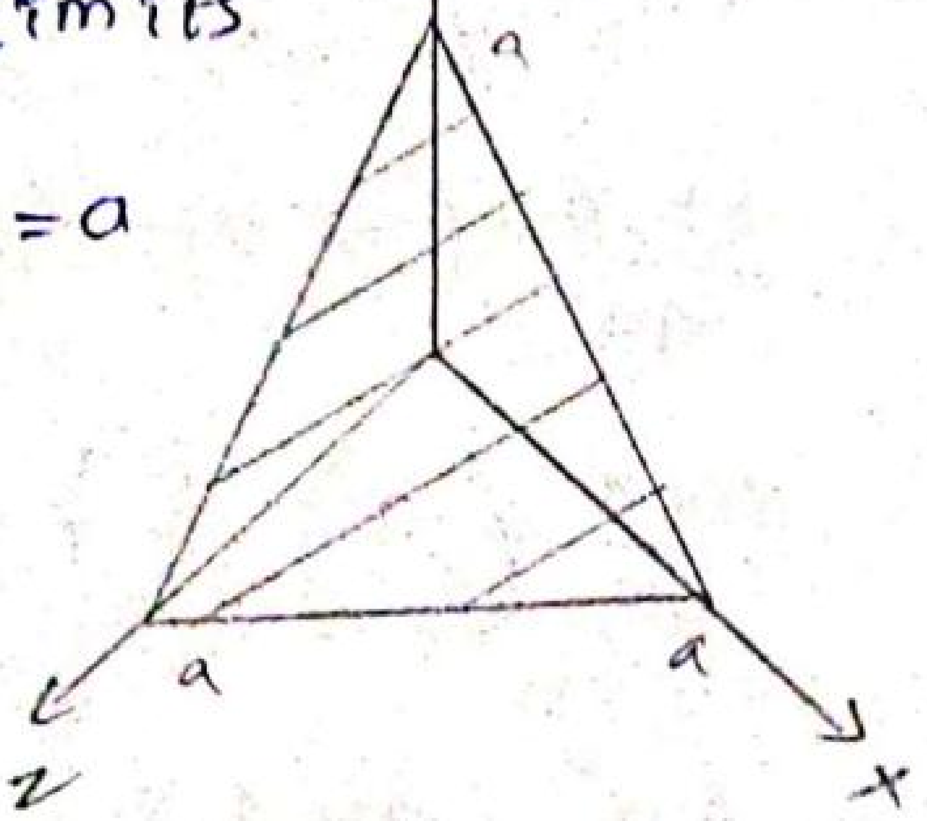
$$= \frac{1}{2}\left(\frac{16-2}{2}\right)$$

$$= \frac{14}{4} \text{ units}$$

1) Calculate the volume of the solid by planes $x=0$, $y=0$, $x+y+z=a$ and $z=0$

Sol: For limits.

$x+y+z=a$



$z = a-x-y$, put $z=0$

$y = a-x$, put $y=0$

$x = a$

'x' varies from $x=0$ to a

'y' varies from $y=0$ to $a-x$

'z' varies from $z=0$ to $a-x-y$

Required volume $= V = \iiint_V dx\,dy\,dz$.

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y} dz\,dy\,dx$$

$$= \int_0^a \int_0^{a-x} (z)_0^{a-x-y}\, dy\,dx$$

$$= \int_{x=0}^{a}\int_0^{a-x} (a-x-y)\,dy\,dx$$

$$= \int_0^a \left[\frac{(a-x-y)^2}{2(-1)}\right]_0^{a-x} dx$$

$$= -\frac{1}{2}\int_0^a \left[((a-x)-(a-x))^2 - (a-x)^2\right] dx$$

$$= \frac{1}{2}\int_0^a (a-x)^2\, dx$$

$$= \frac{1}{2}\left[\frac{(a-x)^3}{3(-1)}\right]_0^a$$

$$= -\frac{1}{6}(0-a^3)$$

$$= \frac{a^3}{6} \text{ units}$$

2) Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Sol: Given,

$$z = +c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

put $z=0$, $y = \pm b\sqrt{1-\frac{x^2}{a^2}}$

put $z=0$ and $y=0$, $\frac{x^2}{a^2} = 1$

$x = \pm a$.

The volume of ellipsoid is 8 times of the portion of it in the first octant (positive).

For which $0 \le x \le a$,

$0 \le y \le b\sqrt{1-\frac{x^2}{a^2}}$ and

$0 \le z \le c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

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$\therefore$ Required volume $= \iiint_V dx\,dy\,dz$

$$= 8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz.dy.dx.$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left[z\right]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx.$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}+\frac{y^2}{b^2}}\; dy\,.dx$$

$$= c\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \quad \text{put } b\sqrt{1-\frac{x^2}{a^2}} = \rho$$

$$b^2\left(1-\frac{x^2}{a^2}\right) = \rho^2$$

$$1-\frac{x^2}{a^2} = \frac{\rho^2}{b^2}.$$

$$= 8c\int_0^a\int_{y=0}^{\rho}\sqrt{\frac{\rho^2}{b^2}-\frac{y^2}{b^2}}\; dy dx.$$

$$= 8\frac{c}{b}\int_0^a\int_0^{\rho}\sqrt{\rho^2-y^2}\; dy dx.$$

$$= 8\frac{c}{b}\int_0^a\left[\frac{y}{2}\sqrt{\rho^2-y^2}+\frac{\rho^2}{2}\sin^{-1}\left(\frac{y}{\rho}\right)\right]_0^{\rho} dx$$

$$= 8\frac{c}{b}\int_0^a\left[0+\frac{\rho^2}{2}\sin^{-1}(1) - (0)\right] dx$$

$$= \frac{8c}{b}\int_0^a \frac{\rho^2}{2}\left(\frac{\pi}{2}\right) dx$$

$$= \frac{8c\pi}{4b}\int_0^a \rho^2 dx.$$

$$= \frac{8c\pi}{4b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right) dx$$

$$= \frac{8c\pi}{4b}.b^2\int_0^a \left(1-\frac{x^2}{a^2}\right) dx.$$

$$= \frac{8bc\pi}{4a^2}\int_0^a (a^2-x^2) dx$$

$$= \frac{8bc\pi}{4a^2}\left(a^2x-\frac{x^3}{3}\right)_0^a$$

$$= \frac{2bc\pi}{a^2}\left(a^3-\frac{a^3}{3}\right) - 0$$

$$= \frac{4a^3}{3}\left(\frac{\cancel{2}bc\pi}{a^2}\right)$$

$$= \frac{4abc\pi}{3} \text{ units}$$

$= 2bc\pi\ \{$

3) obtain the value of sphere $x^2+y^2+z^2 = a^2$. $\left(\frac{4\pi a^3}{3}\right)$

4) Find the volume of tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$

Sol: Given,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$$

i.e., $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

put in x-y plane $z=0$, if $z=0$ then $y = b\left(1-\frac{x}{a}\right)$

If $z=0$ & $y=0$ then $x=a$

$\therefore$ x varies from $x='0'$ to 'a'

y varies from $y='0'$ to $b\left(1-\frac{x}{a}\right)$

z varies from $z=0$ to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$\therefore$ Required volume $= \iiint\limits_V dx\,dy\,dz$

$$= \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx$$

$$= \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} \left[z\right]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\,dx.$$

$$= \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right).dy\,dx$$

put $b\left(1-\frac{x}{a}\right) = t$

$\left(1-\frac{x}{a}\right) = \frac{t}{b}$

$$= c\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} \left(\frac{t}{b}-\frac{y}{b}\right)dy\,dx$$

$$= \frac{1}{b}c\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} (t-y)\,dy\,dx$$

$$= \frac{1}{b}.c\int_0^a \left(ty-\frac{y^2}{2}\right)_0^t dx$$

$$= \frac{1}{b}.c\int_0^a \left(t^2-\frac{t^2}{2}\right)dx$$

$$= \frac{1}{2b}.c\int_0^a t^2\,dx$$

$$= \frac{1}{2b}.c\int_0^a b^2\left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{b}{2}.c\int_0^a \left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)dx$$

$$= \frac{b}{2}.c\left(x+\frac{x^3}{3a^2}-\frac{\cancel{2}x^2}{\cancel{2}a}\right)_0^a$$

$$= \frac{b}{2}.c\left(a+\frac{a^3}{3a^2}-\frac{a^2}{a}\right)$$

$$= c\frac{b}{2}\left(\cancel{a}+\frac{a}{3}-\cancel{a}\right)$$

$$= \frac{abc}{6}\text{ units}$$

3) Given, $x^2+y^2+z^2=a^2$.

$z = \sqrt{a^2-x^2-y^2}$.

put $z=0$, $y=\sqrt{a^2-x^2}$

put $z=0, y=0$, $x=\pm a$.

$\Rightarrow$ The volume of sphere is 8 times of the portion of it in the first octant (positive).

for which $0\le x\le a$,

$0\le y\le\sqrt{a^2-x^2}$ and

$0\le z\le\sqrt{a^2-x^2-y^2}$

$\therefore$ Required volume $= \iiint\limits_V dx\,dy\,dz$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left(z \right)_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sqrt{a^2-x^2-y^2} - 0\right] dy\,dx$$

Let $\sqrt{a^2-x^2} = t$

$a^2-x^2 = t^2$

$$= 8\int_0^a \int_0^t \sqrt{t^2-y^2}\, dy\,dx.$$

$$= 8\int_0^a \left[\frac{y}{t}\sqrt{t^2-y^2} + \frac{t^2}{2}\sin^{-1}\left(\frac{y}{t}\right)\right]_0^t dx.$$

$$= 8\int_0^a \left\{\left[\frac{t}{t}\sqrt{t^2-t^2} + \frac{t^2}{2}\sin^{-1}\left(\frac{t}{t}\right)\right] - [0]\right\} dx$$

$$= 8\int_0^a \left(0 + \frac{t^2}{2}\sin^{-1}(1)\right) dx$$

$$= \frac{8}{2}\int_0^a t^2 . \frac{\pi}{2}\, dx$$

$$= \frac{8\pi}{4}\int_0^a t^2\, dx$$

$$= 2\pi \int_0^a (a^2-x^2)\,dx$$

$$= 2\pi \left(a^2x - \frac{x^3}{3}\right)_0^a$$

$$= 2\pi \left(a^3 - \frac{a^3}{3}\right)$$

$$= \frac{2\pi a^3}{3}(2)$$

$$= \frac{4\pi a^3}{3} \text{ units}$$

Change of Variables in Triple Integrals :-

If (x,y,z) be changing to the variables (u,v,w). By transformation $x = f(u,v,w)$, $y = g(u,v,w)$, $z = h(u,v,w)$

Then,

$$\iiint_{R_{xyz}} F(x,y,z)\, dx\,dy\,dz =$$

$$\iiint_{R_{uvw}} \phi\left(f(u,v,w), g(u,v,w), h(u,v,w)\right) |J|\, du\,dv\,dw.$$

where $|J| = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$

In particatidea ular to change rectangular coordinates (x,y,z) to cylindrical coordinates (ρ,ϕ,z), where $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

and $dx\,dy\,dz = |J|\, d\rho\, d\phi\, dz$.

i.e., $dx\,dy\,dz = \rho\, d\rho\, d\phi\, dz$.

Then, $\iiint_R f(x,y,z)\, dx\,dy\,dz$

$$= \iiint_{R'} f\left(\rho\cos\phi, \rho\sin\phi, z\right) \rho\, d\rho\, d\phi\, dz$$

To change rectangular coordinates (x, y, z) to spherical coordinates (r, θ, ϕ) where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$J\left(\frac{x, y, z}{r, \theta, \phi}\right) = r^2\sin\theta$, and

$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$.

i.e., $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

Then, $\iiint_R f(x, y, z)\,dx\,dy\,dz$

$$= \iiint_{R'} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2\sin\theta\,dr\,d\theta\,d\phi.$$

1) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing into spherical coordinates.

Sol: Put $x = r\sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = r\cos\theta$ and $x^2+y^2+z^2 = r^2$.

Since, the region of integration lies in the first octant $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$, and $0 \le \phi \le \frac{\pi}{2}$.

$$\therefore \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$$

$$= \int_{r=0}^{1}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{1-r^2}}$$

$$= \int_{r=0}^{1} \frac{r^2}{\sqrt{1-r^2}}\,dr \cdot \int_{\phi=0}^{\pi/2} d\phi \cdot \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta$$

$$= -\int_{r=0}^{1} \frac{(1-r^2)-1}{\sqrt{1-r^2}}\,dr \cdot [\phi]_0^{\pi/2} \cdot [-\cos\theta]_0^{\pi/2}$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}\right) dr \cdot \frac{\pi}{2} \times 1$$

$$= \left[\sin^{-1}r - \left(\frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\sin^{-1}r\right)\right]_0^1 \cdot \frac{\pi}{2}.$$

$$= \left[\sin^{-1}(1) - \left(0 + \frac{1}{2}\sin^{-1}(1)\right) - (0)\right]\frac{\pi}{2}$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

2) By transforming into cylindrical coordinates evaluate $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the region $0 \le z \le x^2+y^2 \le 1$.

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introducing cylindrical coordinates $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and $dx\,dy\,dz = \rho\,d\rho\,d\phi\,dz$.

Sol: Given region $0 \le z \le x^2+y^2 \le 1$

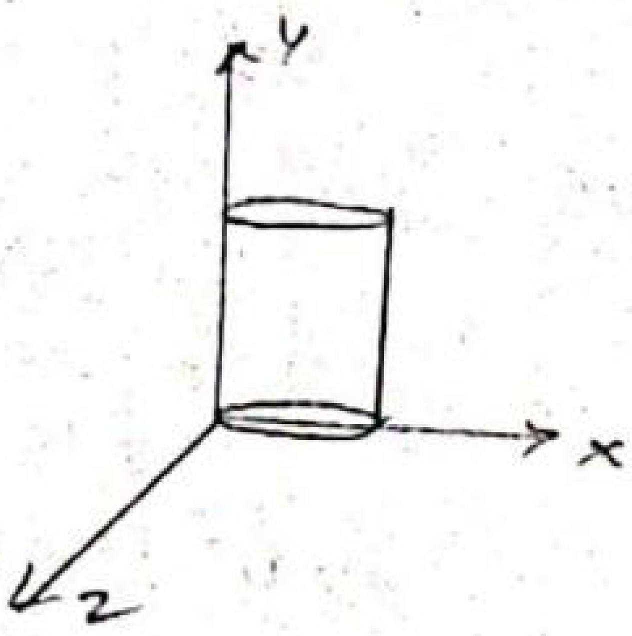
i.e, $0 \le z \le 1$ and

$0 \le x^2+y^2 \le 1$

Here $0 \le \rho^2 \le 1$ (∵ x, y given)

$0 \le \rho \le 1$

And $0 \le \phi \le 2\pi$.



$$\therefore \iiint (x^2+y^2+z^2)\,dx\,dy\,dz =$$

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{1}\int_{z=0}^{1} (\rho^2+z^2)\,\rho\,d\rho\,dz\,d\phi$$

$$= \int_{\phi=0}^{2\pi} d\phi \cdot \int_{\rho=0}^{1}\int_{z=0}^{1} (\rho^3+z^2\rho)\,dz\,d\rho$$

$$= [\phi]_0^{2\pi} \cdot \int_{\rho=0}^{1} \left(\rho^3 z + \frac{z^3}{3}\rho\right)_0^1 d\rho$$

$$= (2\pi) \cdot \int_0^1 \left(\rho^3 + \frac{\rho}{3}\right) d\rho$$

$$= 2\pi \left(\frac{\rho^4}{4} + \frac{\rho^2}{6}\right)_0^1$$

$$= 2\pi \left(\frac{1}{4} + \frac{1}{6}\right)$$

$$= \frac{5\pi}{6}$$

BETA & GAMMA FUNCTIONS:

⇒ Gamma Function:

Algebraic and transendental functions together constitute the elementary functions. Special functions are functions other than the elementary functions such as gamma function, beta function, bessel's function and legendre function e.t.c..

Many integrals which cannot be expressed in terms of elementary functions can be evaluated in terms of beta and gamma functions.

⇒ Gamma Function:

If $n>0$, then the definite integral $\int_0^\infty e^{-x} x^{n-1}\,dx$ then is known as gamma function and is denoted by $\Gamma(n)$.

Thus,

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx}$$

Gamma function is also known as Euler's integral of second kind.

Properties:

1) $\Gamma(1) = 1$

proof: By the definition

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$$

$$\Gamma(1) = \int_0^\infty e^{-x} x^{0}\,dx$$

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx$$

$$\Gamma(1) = \left(-e^{-x}\right)_0^\infty$$

$$\Gamma(1) = \left(-e^{-\infty}+e^{0}\right)$$

$$\Gamma(1) = 0+1$$

$$\Gamma(1) = 1$$

$$\therefore \boxed{\Gamma(1) = 1}$$

Reduction formula for gamma function:

$$\Gamma(n+1) = n\,\Gamma(n)$$

By the definition

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$$

Now, $\Gamma(n+1) = \int_0^\infty e^{-x} x^{n}\,dx$

$$= x^n \int_0^\infty e^{-x}\,dx - \int\left[n x^{n-1} \int_0^\infty e^{-x}\,dx\right]dx$$

$$\left(\because \int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left(f'(x)\int g(x)\,dx\right)dx\right.$$

$$= -\left[x^n e^{-x}\right]_0^\infty + n\int_0^\infty e^{-x} x^{n-1}\,dx$$

$$= -[0-0] + n\,\Gamma(n)$$

$$\therefore \boxed{\Gamma(n+1) = n\,\Gamma(n)}$$

Note:

1) $\Gamma(n) = (n-1)\,\Gamma(n-1)$

2) $\Gamma(n-1) = (n-2)\,\Gamma(n-2)$

3) $\Gamma(n+1) = n!$ where $n = 0,1,2,3,\ldots$

proof: $\Gamma(n+1) = n\,\Gamma(n)$

$$\Gamma(n+1) = n(n-1)\,\Gamma(n-1)$$

$$\Gamma(n+1) = n(n-1)(n-2)\,\Gamma(n-2)$$

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\,\Gamma(n-3)$$

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)(n-4)\,\Gamma(n-4)$$

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)(n-4)\ldots 2.1.\Gamma(1)$$

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)(n-4)\ldots 2.1.$$

$$\Gamma(n+1) = n!$$

$$\therefore \boxed{\Gamma(n+1) = n!}$$

1) Prove that $\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$

Sol: By the definition $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1}\,dt$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2}\,dt$$

put $t = x^2$ then $dt = 2x\,dx$.

when $t = 0$, $x = 0$

$t = \infty$, $x = \infty$

$$\therefore \Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x^2}\left(+x^2\right)^{-1/2} 2x\,dx$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-x^2} x^{-1}\cdot x\,dx$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\,dx$$

$\therefore \Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-x^2}dx \rightarrow (1)$

Replace 'y' for 'x' in equ (1) we get.

$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-y^2}dy \rightarrow (2)$

$(1)\times(2) \Rightarrow \Gamma(\frac{1}{2})\Gamma(\frac{1}{2}) = 4\int_0^\infty e^{-x^2}dx \cdot \int_0^\infty e^{-y^2}dy$

$$\left[\Gamma(\tfrac{1}{2})\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$$

Change to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$.

then $dxdy = rdrd\theta$.

The region of integration in first quadrant.

'θ' varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$

'r' varies from $r = 0$ to $r = \infty$

$$\therefore \left[\Gamma(\tfrac{1}{2})\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} rdrd\theta$$

$$= -2\int_0^{\pi/2}\int_0^\infty e^{-r^2} -2rdrd\theta$$

$$= -2\int_0^{\pi/2}\left(e^{-r^2}\right)_0^\infty d\theta$$

$$= -2\int_0^{\pi/2}(e^{-\infty} - e^0)d\theta$$

$$= -2\int_0^{\pi/2}(-1)d\theta$$

$$= 2\int_0^{\pi/2}d\theta$$

$$= 2(\theta)_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi$$

$$\therefore \left[\Gamma(\tfrac{1}{2})\right]^2 = \pi$$

$$\boxed{\Gamma(\tfrac{1}{2}) = \sqrt{\pi}}$$

Hence proved

$\rightarrow$ If 'n' is a negative fraction then

$$\boxed{\Gamma(n) = \frac{\Gamma(n+K+1)}{n(n+1)(n+2)\cdots(n+K)}}$$ where $n \neq 0, -1, -2, \ldots$

where 'K' is least positive integer. $n+K+1 > 0$

Note: $\Gamma(n)$ does not exist for $n = 0, -1, -2, -3, -4, \ldots$

1) Evaluate the following using Gamma function.

(i) $\Gamma(7)$

Sol: $= 6!$

$= 720$

(ii) $\Gamma\left(\frac{9}{2}\right)$

Sol: $= \left(\frac{9}{2} - 1\right)\Gamma\left(\frac{9}{2} - 1\right)$

$= \frac{7}{2} \Gamma\left(\frac{7}{2}\right)$

$= \frac{7}{2}\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$

$= \frac{7}{2}\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)$

$= \frac{7}{2}\times\frac{5}{2}\times\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)$

$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\Gamma\left(\frac{3}{2}\right)$

$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$

$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}$

$\left(\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right)$

$= \frac{105}{16}\sqrt{\pi}.$

(iii) $\Gamma\left(\frac{11}{3}\right)$

Sol: $\Gamma\left(\frac{11}{3}\right) = \frac{8}{3}\times\frac{5}{3}\times\frac{2}{3}\times\Gamma\left(\frac{2}{3}\right)$

$= \frac{80}{27}\Gamma\left(\frac{2}{3}\right)$

(iv) $\Gamma(10)$

Sol: $9!$

(v) $\Gamma\left(-\frac{3}{2}\right)$

Sol: Here $n = -\frac{3}{2}$

$n+K+1 > 0$

$-\frac{3}{2}+K+1 > 0$

$K > \frac{1}{2}$

Here $K = 1$

$\therefore n = -\frac{3}{2}, K = 1$

$\Gamma\left(-\frac{3}{2}\right) = \dfrac{\Gamma\left(-\frac{3}{2}+1+1\right)}{-\frac{3}{2}\left(-\frac{3}{2}+1\right)}$

$= \dfrac{\Gamma\left(\frac{1}{2}\right)}{-\frac{3}{2}\times-\frac{1}{2}}$

$= \frac{4\sqrt{\pi}}{3}$

(vi) $\Gamma(-2.5)$

Sol: Here $n = -2.5 = -\frac{5}{2}$

$\therefore n+K+1 > 0$

$-\frac{5}{2}+K+1 > 0$

$K > \frac{3}{2}$

$\therefore K = 2.$

$\Gamma\left(-\frac{5}{2}\right) = \dfrac{\Gamma\left(-\frac{5}{2}+2+1\right)}{-\frac{5}{2}\times\left(-\frac{5}{2}+1\right)\times\left(-\frac{5}{2}+2\right)}$

$\Gamma\left(-\frac{5}{2}\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)}{-\frac{5}{2}\times-\frac{3}{2}\times-\frac{1}{2}}$

$\Gamma\left(-\frac{5}{2}\right) = -\frac{8\sqrt{\pi}}{15}$

(vii) $\Gamma\left(-\frac{1}{2}\right)$

Sol: By using reduction formula

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

Here $n = -\frac{1}{2}$

$\therefore \Gamma\left(-\frac{1}{2}\right) = \dfrac{\Gamma\left(-\frac{1}{2}+1\right)}{-1/2}$

$$= \frac{-2\Gamma(\frac{1}{2})}{\cancel{-2\sqrt{\pi}}}$$

$$= -2\sqrt{\pi}$$

2) Evaluate the following.

(i) $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$

Sol: Let $x^3 = y$

$x = y^{1/3}$

Then $dx = \frac{1}{3} \cdot y^{1/3-1} \cdot dy$.

when $x = 0, \; y = 0$

$x = \infty, \; y = \infty$

$$\therefore \int_0^\infty x^{1/2} \cdot e^{-x^3} dx$$

$$= \int_0^\infty (y^{1/3})^{1/2} \cdot e^{-y} \cdot \frac{1}{3} y^{1/3-1} dy.$$

$$= \frac{1}{3}\int_0^\infty y^{1/6} \cdot e^{-y} \cdot y^{1/3-1} dy$$

$$= \frac{1}{3}\int_0^\infty e^{-y} \cdot y^{\frac{1}{2}-1} dy$$ It is in standard form

$$= \frac{1}{3}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{3}$$

(ii) $\int_0^\infty x^7 \cdot e^{-2x^2} dx$

Sol: Let

$2x^2 = y$

$x^2 = \frac{y}{2}$

$x = \frac{y^{1/2}}{2^{1/2}}$

Then $dx = \frac{1}{2^{1/2}} \cdot \frac{1}{2y^{1/2}} \cdot dy$

$= \int_0$ when $x = 0, \; y = 0$

$x = \infty, \; y = \infty$

$$\therefore \int_0^\infty x^7 \cdot e^{-2x^2} dx = \int_0^\infty \left(\frac{y^{1/2}}{2^{1/2}}\right)^7 \cdot e^{-2(\frac{y}{2})} \cdot \frac{1}{2^{1/2} \cdot 2} y^{-1/2} \cdot dy$$

$$= \frac{1}{2^5}\int y^{7/2} \cdot e^{-y} \cdot y^{-1/2} dy$$

$$= \frac{1}{2^5}\int e^{-y} y^3 dy$$

$$= \frac{1}{2^5}\int e^{-y} \cdot y^{4-1} dy$$

$$= \frac{1}{2^5}\Gamma(4)$$

$$= \frac{3!}{2^5}$$

(iii) $\int_0^\infty e^{-x^2} \cdot x^{7/2} dx$.

Sol: $x^2 = y$

$x = y^{1/2}$

Then, $dx = \frac{1}{2\sqrt{y}} dy$

when $x = 0$, $y = 0$
$x = \infty$, $y = \infty$

$$\therefore \int_0^\infty e^{-x^2} x^{7/2} dx = \int_0^\infty e^{-y} (y^{1/2})^{7/2} \cdot \frac{1}{2} y^{-1/2} dy$$

$$= \frac{1}{2} \int_0^\infty e^{-y} y^{7/4} \cdot y^{-1/2} dy$$

$$= \frac{1}{2} \int_0^\infty e^{-y} y^{5/4} dy$$

$$= \frac{1}{2} \int_0^\infty e^{-y} y^{\frac{9}{4}-1} dy$$

$$= \frac{1}{2} \Gamma\left(\frac{9}{4}\right)$$

$$= \frac{1}{2} \left(\frac{5}{4}\right)\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{4}\right)$$

$$= \frac{5}{32} \cdot \Gamma\left(\frac{1}{4}\right)$$

(iv) $\int_0^\infty e^{-x^2} \cdot x^4 dx$. $\underline{\underline{\text{Ans}}} = \frac{3}{8}\sqrt{\pi}$

(v) $\int_0^\infty e^{-4x} \cdot x^{3/2} dx$. $\underline{\underline{\text{Ans}}}: \frac{3}{4^{7/2}} \cdot \sqrt{\pi}$

(iv) $x^2 = y$
$x = y^{1/2}$
$dx = \frac{1}{2} y^{-1/2} dy$

limits, $x = 0$, $y = 0$
$x = \infty$, $y = \infty$

$$\therefore \int_0^\infty e^{-x^2} \cdot x^4 dx = \int_0^\infty e^{-y} (y^{1/2})^4 \cdot \frac{1}{2} y^{-1/2} dy$$

$$= \frac{1}{2} \int_0^\infty e^{-y} y^2 \cdot y^{-1/2} dy$$

$$= \frac{1}{2} \int_0^\infty e^{-y} (y)^{3/2} dy$$

$$= \frac{1}{2} \int_0^\infty e^{-y} (y)^{\frac{5}{2}-1} dy$$

$$= \frac{1}{2} \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{1}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{8} \sqrt{\pi}$$

(v) Let $4x = y$
$x = \frac{y}{4}$
$dx = \frac{dy}{4}$

limits, when $x = 0$, $y = 0$
$x = \infty$, $y = \infty$

$$\therefore \int_0^\infty e^{-4x} (x)^{3/2} \cdot dx$$

$$= \int_0^\infty e^{-y} \left(\frac{y}{4}\right)^{3/2} \cdot \frac{dy}{4}$$

$$= \left(\frac{1}{4}\right)^{5/2} \int_0^\infty e^{-y} (y)^{3/2} dy$$

$$= \left(\frac{1}{4}\right)^{5/2} \int_0^\infty e^{-y} y^{\left(\frac{5}{2}-1\right)} dy$$

$$= \left(\frac{1}{4}\right)^{5/2} \cdot \Gamma\left(\frac{5}{2}\right)$$

$$= \left(\frac{1}{4}\right)^{5/2} \cdot \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)$$

$$= \left(\frac{1}{4}\right)^{7/2} \times \frac{3}{4} \sqrt{\pi}$$

$$= \frac{3}{(4)^{7/2}} \sqrt{\pi}$$

(vi) $\int_0^\infty a^{-bx^2} dx$.

Sol: Given, $\int_0^\infty a^{-bx^2} dx$

$= \int_0^\infty e^{\log_e a^{-bx^2}} dx$

$= \int_0^\infty e^{-bx^2 \log a} dx$

Let $bx^2 \log a = y$

$x^2 = \frac{y}{b\log a}$

$x = \frac{y^{1/2}}{b^{1/2}(\log a)^{1/2}}$

Then $dx = \frac{1}{2y^{1/2}} \cdot \frac{1}{(b\log a)^{1/2}} dy$.

limits $\Rightarrow$ when $x = 0, y = 0$
$x = \infty, y = \infty$

$\therefore \int_0^\infty a^{-bx^2} dx = \int_0^\infty e^{-y} \cdot \frac{1}{2} y^{-1/2} \frac{1}{(b\log a)^{1/2}} dy$

$= \frac{1}{2(b\log a)^{1/2}} \int_0^\infty e^{-y} \cdot y^{-1/2} dy$

$= \frac{1}{2(b\log a)^{1/2}} \int_0^\infty e^{-y} y^{\frac{1}{2}-1} dy$

$= \frac{1}{2(b\log a)^{1/2}} \Gamma\left(\frac{1}{2}\right)$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{b\log a}} \cdot \sqrt{\pi}$ //

(vii) $\int_0^\infty 3^{-4x^2} dx$

Sol: ($\because$ check previous one)

$\therefore \int_0^\infty a^{-bx^2} dx = \int_0^\infty 3^{-4x^2} dx$

$a = 3, b = 4$

$= \frac{1}{2} \frac{1}{\sqrt{4\log 3}} \cdot \sqrt{\pi}$ //

(3) Express $\int_0^\infty \frac{x^c}{c^x} \cdot dx \; (c>1)$ in terms of Γ-function.

Sol: Given, $\int_0^\infty \frac{x^c}{c^x} dx$.

$= \int_0^\infty c^{-x} \cdot x^c dx$

$= \int_0^\infty e^{\log_e c^{-x}} x^c dx$

$= \int_0^\infty e^{-x\log c} x^c dx$

Let $x\log c = y$

$x = \frac{y}{\log c}$

Then $dx = \frac{1}{\log c} dy$

when $x = 0, y = 0$
$x = \infty, y = \infty$

$\therefore \int_0^\infty e^{-x\log c} \cdot x^c dx$

$$= \int_0^\infty e^{-y} \cdot \frac{y^c}{(\log c)^c} \cdot \frac{1}{\log c}\, dy$$

$$= \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-y} y^c\, dy$$

$$= \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-y} y^{(c+1-1)}\, dy$$

$$= \frac{1}{(\log c)^{c+1}} \cdot \Gamma(c+1)$$

4) $\int_0^\infty \frac{x^4}{4^x}\, dx$

Sol: By previous question

$c = 4$

$$\Rightarrow \int_0^\infty 4^{-x} \cdot x^4\, dx = \int_0^\infty c^{-x} x^c\, dx.$$

$$= \frac{1}{(\log 4)^{4+1}} \Gamma(4+1)$$

$$= \frac{4!}{(\log 4)^{4+1}}$$

5) Show that $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma(n)$

Sol: $= \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx$

Let $\log\left(\frac{1}{x}\right) = y$

$\frac{1}{x} = e^y$

$x = e^{-y}$

Then $dx = -e^{-y} dy$

when $x=0$, $y=\infty$

$x=1$, $y=0$

$$\therefore \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx = \int_\infty^0 y^{n-1} \cdot -e^{-y} dy$$

$$= \int_0^\infty e^{-y} y^{n-1} dy$$

$$= \Gamma(n)$$

6) Show that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$

where $p, q > 0$

Sol: Let $\log\frac{1}{y} = x$

$\frac{1}{y} = e^x$

$y = e^{-x}$

Then, $dy = -e^{-x} dx$

when $y=0$, $x=\infty$

$y=1$, $x=0$

$$\therefore \int_0^1 y^{q-1} \left(\log\frac{1}{y}\right)^{p-1} dy = \int_\infty^0 (e^{-x})^{q-1} (x)^{p-1} \cdot -e^{-x} dx$$

$$= \int_0^\infty e^{-xq} x^{p-1} dx.$$

Let $xq = t$

$x = \frac{t}{q}$

Then, $dx = \frac{dt}{q}$

Limits when $x=0,\ t=0$
$x\to\infty,\ t\to\infty$

$$\therefore \int_0^\infty e^{-ax}\,x^{p-1}\,dx$$

$$= \int_0^\infty e^{-t}\left(\frac{t}{a}\right)^{p-1}\cdot\frac{dt}{a}$$

$$= \frac{1}{a^p}\int_0^\infty e^{-t}\,t^{p-1}\,dt$$

$$= \frac{1}{a^p}\,\Gamma(p)$$

$$= \frac{\Gamma(p)}{a^p}$$

⇒ Prove that $\int_0^1 x^m(\log x)^n\,dx$

$$= \frac{(-1)^n\,n!}{(m+1)^{n+1}}$$ where $n\in Z$

and $m>-1$. Hence evaluate

$$\int_0^1 x(\log x)^3\,dx$$

Sol: Given, $\int_0^1 x^m(\log x)^n\,dx$

Let $\log\frac{1}{x}=y$

$$-\log x = y$$
$$\log x = -y$$
$$x = e^{-y}$$

Then $dx = -e^{-y}dy$

when $x=0$, then $y\to\infty$

$x=1$, then $y\to 0$

$$\int_0^1 x^m(\log x)^n\,dx = \int_\infty^0 (e^{-y})^m(-y)^n\,(-e^{-y})\,dy$$

$$= \int_0^\infty e^{-my}\,y^n\,e^{-y}\,dy$$

$$= (-1)^n\int_0^\infty e^{-y(m+1)}\,y^n\,dy$$

Let $y(m+1)=t$

$$y = \frac{t}{m+1}$$

$$dy = \frac{dt}{m+1}$$

when $y=0,\ t=0$
$y=\infty,\ t=\infty$

$$\therefore (-1)^n\int_0^\infty e^{-y(m+1)}\,y^n\,dy$$

$$= (-1)^n\int_0^\infty e^{-t}\left(\frac{t}{m+1}\right)^n\cdot\frac{dt}{(m+1)}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-t}\,t^n\,dt$$

$$= (-1)^n\cdot\frac{1}{(m+1)^{n+1}}\int_0^\infty e^{-t}\,t^{(n+1)-1}\,dt$$

$$= (-1)^n\cdot\frac{1}{(m+1)^{n+1}}\,\Gamma(n+1)$$

$$= (-1)^n\cdot\frac{1}{(m+1)^{n+1}}\cdot n!$$

Hence proved

Given, $\int_0^1 x(\log x)^5 dx$

Here $m=1$, $n=5$

$\therefore (-1)^3\left(\frac{1}{(1+1)^{3+1}}\right) \cdot 3!$

$= \frac{-6}{2^4}$

$= \frac{-6}{16}$

$= \frac{-3}{8}$

8) Show that $\int_0^1 \sqrt[3]{x \log \frac{1}{x}}\, dx$

$= \left(\frac{3}{4}\right)^{4/3} \Gamma\left(\frac{4}{3}\right)$

Sol: $\int x^{1/3}\left(\log\left(\frac{1}{x}\right)\right)^{1/3} dx$

$\log \frac{1}{x} = y$

$-\log x = y$

$\log x = -y$

$x = e^{-y}$

$dx = e^{-y} dy$

Limits

when $x=0$, $y=\infty$

$x=1$, $y=0$

$= \int_{\infty}^{0} \left(-e^{-y}\right)^{1/3} (y)^{1/3} e^{-y} dy$

$= \int_0^{\infty} \left(e^{-y}\right)^{4/3} (y)^{1/3} dy$

$y\left(\frac{4}{3}\right) = t$

$y = \frac{3}{4} t$

$dy = \frac{3}{4} dt$

Limits

when $x=0$, $y=0$

$x=\infty$, $y=\infty$

$\therefore \int_0^{\infty} \left(e^{-y}\right)^{4/3} y^{1/3} dy$

$= \int_0^{\infty} e^{-t}\left(\frac{3}{4}t\right)^{1/3} \frac{3}{4} dt$

$= \left(\frac{3}{4}\right)^{4/3} \int_0^{\infty} e^{-t}(t)^{1/3} dt$

$= \left(\frac{3}{4}\right)^{4/3} \int_0^{\infty} e^{-t}(t)^{4/3-1} dt$

$= \left(\frac{3}{4}\right)^{4/3} \Gamma\left(\frac{4}{3}\right)$

9) Express $\int_0^{\infty} e^{-ax} \cdot x^{m-1} \cdot \sin bx\, dx$ in terms of Γ function.

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Sol: Now, $\int_0^\infty e^{-ax} x^{m-1} dx$

Let $ax = y$

$x = \frac{y}{a}$

Then $dx = \frac{dy}{a}$

limits, when $x = 0, y = 0$

$x = \infty, y = \infty$

$$\therefore \int_0^\infty e^{-ax} \cdot x^{m-1} dx = \int_0^\infty e^{-y} \left(\frac{y}{a}\right)^{m-1} \frac{dy}{a}$$

$$= \frac{1}{a^m} \int_0^\infty e^{-y} y^{m-1} dy$$

$$= \frac{1}{a^m} \cdot \Gamma(m)$$

$$\therefore \int_0^\infty e^{-ax} x^{m-1} dx = \frac{1}{a^m} \cdot \Gamma(m) \rightarrow (1)$$

$$= \int_0^\infty e^{-ax} \cdot x^{m-1} \cdot \sin bx \, dx$$

$$= \int_0^\infty e^{-ax} \cdot x^{m-1} \operatorname{Img}(e^{ibx}) dx$$

$$(\because e^{ibx} = \cos bx + i \sin bx)$$

$$= \operatorname{Img}\left(\int_0^\infty e^{-x(a-ib)} \cdot x^{m-1} dx\right)$$

$$= \operatorname{Img}\left[\frac{1}{(a-ib)^m} \cdot \Gamma(m)\right]$$

$(\because (1))$

Let $a = r\cos\theta$, $b = r\sin\theta$

$$= \operatorname{Img}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^m} \Gamma(m)\right]$$

$$= \operatorname{Img}\left[\frac{1}{r^m(\cos\theta - i\sin\theta)^m} \Gamma(m)\right]$$

$$= \operatorname{Img}\left[\frac{1}{r^m} \cdot \frac{1}{\cos m\theta - i\sin m\theta} \Gamma(m)\right]$$

$$= \operatorname{Img}\left[\frac{1}{r^m} \cdot \frac{(\cos m\theta + i\sin m\theta)}{1} \cdot \Gamma(m)\right]$$

$$= \operatorname{Img} \frac{\sin m\theta}{r^m} \cdot \Gamma(m)$$

$$\therefore \int_0^\infty e^{-ax} x^{m-1} \sin bx \, dx = \frac{\sin m\theta}{r^m} \Gamma(m)$$

⇒ Beta Function:

If $m, n > 0$ then the definite integral

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx$$

is called Beta function and is denoted by $\beta(m,n)$ and is defined as

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx}$$

β-function is also known the Euler or Eulerian integral of first kind

Properties :-

1) $\beta(m,n) = \beta(n,m)$ it is symmetric property of β- function.

Proof :-

$$\beta(m,n) = \int_0^1 x^{m-1}\cdot(1-x)^{n-1}dx.$$

$$\left(\because \int_0^a f(x)dx = \int_0^a f(a-x)\,dx\right)$$

$$\beta(m,n) = \int_0^1 (1-x)^{m-1}(1-(1-x))^{n-1}dx$$

$$= \int_0^1 (1-x)^{m-1}x^{n-1}dx.$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx$$

$$= \beta(n,m)$$

$$\therefore \beta(m,n) = \beta(n,m)$$

2) $\beta(m,n+1)+\beta(m+1,n) = \beta(m,n)$

By the definition

$$\beta(m,n) = \int_0^1 x^{m-1}\cdot(1-x)^{n-1}dx$$

Now, $\beta(m,n+1)+\beta(m+1,n)$

$$= \int_0^1 x^{m-1}\cdot(1-x)^{(n+1)-1}dx + \int_0^1 x^{m+1-1}(1-x)^{n-1}dx$$

$$= \int_0^1 x^{m-1}\cdot(1-x)^{n}dx + \int_0^1 x^{m}(1-x)^{n-1}dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(1-x+x)\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= \beta(m,n)$$

Beta functions in terms of Trigonometric functions :-

By the definition of

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$x = \sin^2 y$$

$$dx = 2\sin y\cos y\,dy.$$

when $x=0,\ y=0$

$x=1,\ y = \frac{\pi}{2}$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= \int_0^{\pi/2} (\sin^2 y)^{m-1}(1-\sin^2 y)^{n-1}2\sin y\cos y\,dy$$

$$= 2\int_0^{\pi/2} \sin y^{2m-2}\cos y^{2n-2}\sin y\cos y\,dy$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}y\cdot\cos^{2n-1}y\,dy$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}y\cdot\cos^{2n-1}y\,dy} \rightarrow (1)$$

Corollary :-

$$\int_0^{\frac{\pi}{2}} \sin^p y\cos^q y = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Put $2m-1=p$, $2n-1=q$

$m=\frac{p+1}{2}$, $n=\frac{q+1}{2}$

put m,n in ①, we get

$$\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)=2\int_0^{\pi/2}\sin^p y\cdot\cos^q y\,dy$$

$$\boxed{\therefore \int_0^{\pi/2}\sin^p y\cos^q y\,dy=\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)}$$

Other forms of β-function:

Form 1: Show that β

$$\beta(p,q)=\int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\cdot dy=\int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}dy$$

(or) $\beta(m,n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

By the definition

$$\beta(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

put $x=\frac{1}{1+y}$ then $dx=\frac{-1}{(1+y)^2}dy$

limits, when $x=0$, $y\to\infty$

$x=1$, $y=0$

$$\beta(p,q)=\int_\infty^0\left(\frac{1}{1+y}\right)^{p-1}\left(1-\frac{1}{1+y}\right)^{q-1}\frac{-1}{(1+y)^2}dy$$

$$=\int_0^\infty\left(\frac{1}{1+y}\right)^{p-1}\left(\frac{y}{1+y}\right)^{q-1}\frac{1}{(1+y)^2}dy$$

$$=\int_0^\infty \frac{y^{q-1}}{(1+y)^{p-1+q-1+2}}dy$$

$$=\int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy$$

$$\therefore \beta(p,q)=\int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy$$

$$\beta(p,q)=\int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}dy$$

$(\because \beta(p,q)=\beta(q,p))$

Form 2: show that

$$\beta(p,q)=\int_0^1\frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx.$$

By form ①,

$$\beta(p,q)=\int_0^\infty\frac{x^{q-1}}{(1+x)^{p+q}}dx$$

$$\beta(p,q)=\int_0^1\frac{x^{q-1}}{(1+x)^{p+q}}+\int_1^\infty\frac{x^{q-1}}{(1+x)^{p+q}}dx$$

Put $x=\frac{1}{y}$ in second integral

then $dx=\frac{-1}{y^2}dy$

when $x=1$, $y=1$

$x=\infty$, $y=0$

$$\therefore \beta(p,q)=\int_0^1\frac{x^{q-1}}{(1+x)^{p+q}}dx+\int_1^0\frac{\left(\frac{1}{y}\right)^{q-1}}{\left(1+\frac{1}{y}\right)^{p+q}}\frac{-1}{y^2}dy$$

$$= \int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}}dx + \int_0^1 \frac{y^{p+q}}{y^{q-1}(1+y)^{p+q}} \cdot \frac{1}{y^2}dy$$

$$= \int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}}dx + \int_0^1 \frac{y^{p-1}}{(1+y)^{p+q}}dy$$

$$= \int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}}dx + \int_0^1 \frac{x^{p-1}}{(1+x)^{p+q}}dx$$

$$= \int_0^1 \frac{x^{q-1}+x^{p-1}}{(1+x)^{p+q}}dx$$

Form 3: Show that

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}dx = \frac{1}{a^n b^m}\beta(m,n)$$

Sol: $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}dx$

$$= \int_0^\infty \frac{x^{m-1}}{a^{m+n}\left(1+\frac{b}{a}x\right)^{m+n}}dx$$

Let $\frac{b}{a}x = y \Rightarrow x = \frac{a}{b}y$

$dx = \frac{a}{b}dy$

$$= \int_0^\infty \frac{((a/b)y)^{m-1}}{a^{m+n}(1+y)^{m+n}} \cdot \frac{a}{b}y.$$

$$= \frac{1}{a+m} \cdot \left(\frac{a}{b}\right)^{m-1} \cdot \left(\frac{a}{b}\right)^1 \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

$$= \frac{1}{a^n b^m} \cdot \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

$$= \frac{1}{a^n b^m} \cdot \beta(m,n) \quad (\because \text{form I})$$

Hence proved

Show that $\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx$

$= (b-a)^{m+n-1} \cdot \beta(m,n)$

Sol: Let $(x-a) = (b-a)y$

$x = (b-a)y + a$

Then $dx = (b-a)dy$

limits, when $x = a, y = 0$

$x = b, y = 1$

$$\therefore \int_a^b (x-a)^{m-1}(b-x)^{n-1}dx$$

$$= \int_0^1 [(b-a)y]^{m-1}\left[b-[(b-a)y+a]\right]^{n-1}(b-a)dy$$

$$= \int_0^1 (b-a)^{m-1} \cdot y^{m-1}\left((b-a)-(b-a)y\right)^{n-1}(b-a)dy$$

$$= \int_0^1 (b-a)^{m-1}y^{m-1}(b-a)^{n-1} \cdot (1-y)^{n-1}(b-a)dy$$

$$= (b-a)^{m-\cancel{1}+n-1+\cancel{1}}\int_0^1 y^{m-1}(1-y)^{n-1}dy$$

$$= (b-a)^{m+n-1} \beta(m,n)$$

Hence proved.

St: **Relation between Beta and Gamma functions** $\beta(m,n) = \frac{\beta(m) \cdot \beta(n)}{\Gamma(m+n)}$

Proof: By the definition of Γ-function

$$\Gamma(m) = \int_0^\infty e^{-t} \cdot t^{m-1}dt$$

put $t = x^2$ then $dt = 2x\,dx$

when $t = 0, x = 0$

$t = \infty, x = \infty$

$$\therefore \Gamma(m) = \int_0^\infty e^{-t}\cdot t^{m-1}\,dt = \int_0^\infty e^{-x^2}(x^2)^{m-1}\,2x\,dx$$

$$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-2}.\,x\,dx$$

$$\Gamma(m) = 2\int_0^\infty e^{-x^2}.\,x^{2m-1}\,dx$$

$$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \rightarrow (1)$$

Similarly, $2\int_0^\infty e^{-y^2} y^{2n-1}\,dy \rightarrow (2)$

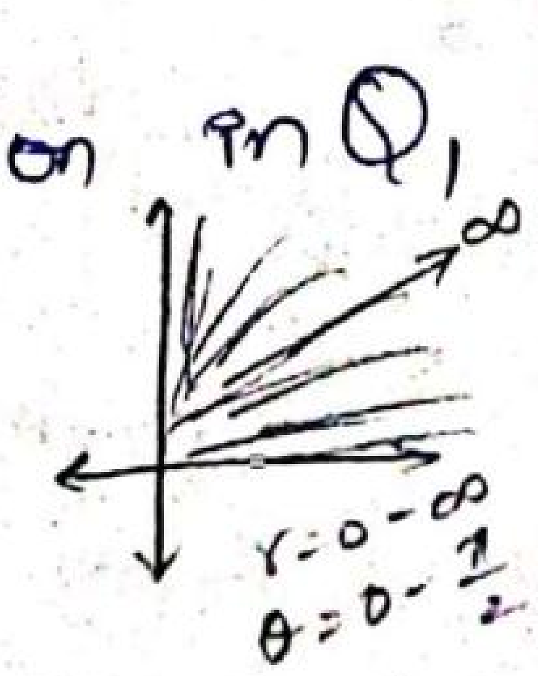
Multiply (1) and (2) we get:

$$\Gamma(m).\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dx\,dy$$

Changing to polar coordinates by putting $x = r\cos\theta,\ y = r\sin\theta$,

$x^2+y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$.

The region of integration in Q_1



'θ' varies from

$\theta = 0$ to $\theta = \frac{\pi}{2}$

'r' varies from

$r=0$ to $r=\infty$

$$\therefore \Gamma(m).\Gamma(n) = 4.\int_0^{\pi/2}\int_0^\infty e^{-r^2}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}\,r\,dr\,d\theta.$$

$$= 2\int_0^{\pi/2}(\cos\theta)^{2m-1}(\sin\theta)^{2n-1}\,d\theta\ \cancel{d\theta}\ \times$$

$$2\int_0^\infty e^{-r^2} r^{2m-1+2n-1+1}\,dr$$

$$= 2\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta\,.\,2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\,dr$$

$$= \beta(m,n).\Gamma(m+n)$$

($\because$ (1), (2) in trigonometric functions)

$$\therefore \Gamma(m).\Gamma(n) = \beta(m,n).\Gamma(m+n)$$

$$\boxed{\beta(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}}$$

→ Given that $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$

show that $\Gamma(n).\Gamma(1-n) = \frac{\pi}{\sin n\pi}$. $(0<n<1)$ Hence evaluate $\int_0^\infty \frac{dy}{1+y^4}$.

Sol: By form-I, $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} = \beta(m,n)$

$$\therefore \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

$$\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Put $m+n=1 \Rightarrow m=1-n$

$$\frac{\Gamma(1-n).\Gamma(n)}{\Gamma(1)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^1}\,dx.$$

$$\therefore \Gamma(n).\Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx \rightarrow (1)$$

It is given that

$$\int_0^\infty \frac{x^{n-1}}{1+x} = \frac{\pi}{\sin n\pi}$$

from ① $\boxed{\Gamma(n)\cdot\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$

Now, $\int_0^\infty \frac{1}{1+y^4}\,dy$

Put $x = y^4 \Rightarrow y = x^{1/4}$

Then $dy = \frac{1}{4}x^{1/4-1}\,dx$

$$\int_0^\infty \frac{1}{1+y^4}\,dy = \int_0^\infty \frac{1}{1+x}\cdot\frac{1}{4}x^{\frac{1}{4}-1}\,dx$$

$$= \frac{1}{4}\int_0^\infty \frac{x^{1/4-1}}{(1+x)^1}\,dx$$

$$= \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{4}\right)$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

$$= \frac{1}{4}\frac{\Gamma\left(1-\frac{1}{4}\right)\cdot\Gamma\left(\frac{1}{4}\right)}{1}$$

$$= \frac{1}{4}\cdot\frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}\pi}{4}$$

$$= \frac{\pi}{2\sqrt{2}} //$$

→ $\int_0^2 x\sqrt[3]{8-x^3}\,dx$

Sol: $= \int_0^2 x\sqrt[3]{8-x^3}\,dx$

$$= \int_0^2 x\cdot 2\left(1-\frac{x^3}{8}\right)^{1/3}dx$$

Let $\frac{x^3}{8} = y$

$x^3 = 8y$

$x = 2y^{1/3}$

Then $dx = \frac{2}{3}y^{1/3-1}\,dy$

when $x=0, y=0$ and $x=2, y=1$

$$\therefore \int_0^1 2\cdot y^{1/3}\cdot 2(1-y)^{1/3}\cdot\frac{2}{3}y^{1/3-1}\,dy$$

$$= \frac{8}{3}\int_0^1 y^{\frac{2}{3}-1}(1-y)^{1/3}\,dy$$

$$= \frac{8}{3}\int_0^1 y^{\frac{2}{3}-1}(1-y)^{\frac{4}{3}-1}\,dy$$

$$= \frac{8}{3}\beta\left(\frac{2}{3},\frac{4}{3}\right)$$

$$= \frac{8}{3}\cdot\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{6}{3}\right)} \quad \left(\tfrac{4}{3}-1\right)\Gamma($$

$$= \frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\cdot\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{\Gamma(2)}$$

$$= \frac{8}{9}\frac{\Gamma\left(1-\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)}{1}$$

$$= \frac{8}{9}\cdot\frac{\pi}{\sin\frac{\pi}{3}}$$

$$= \frac{8}{9}\cdot\frac{\pi}{\sqrt{3}/2}$$

$$= \frac{16\pi}{9\sqrt{3}} //$$

→ $\int_0^{\pi/2} \sin^2\theta\cdot\cos^4\theta\cdot d\theta$

Sol: $\int_0^{\frac{\pi}{2}} \sin^2\theta\cdot\cos^4\theta\cdot d\theta = \frac{1}{2}\beta\left(\frac{2+1}{2},\frac{4+1}{2}\right)$

$$= \frac{1}{2}\beta\left(\frac{3}{2},\frac{5}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma(\frac{3}{2}) \cdot \Gamma(5/2)}{\Gamma(4)}$$

$$= \cancel{\frac{1}{2 \cdot 6}} \; \frac{\frac{1}{2} \cdot \frac{1}{2} \Gamma(\frac{1}{2}) \times \frac{3}{2} \times \frac{1}{2} \Gamma(\frac{1}{2})}{\Gamma(4)}$$

$$= \frac{1}{\cancel{12}2} \cdot \cancel{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \sqrt{\pi} \cdot \sqrt{\pi}}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \sqrt{\pi}}{6}$$

$$= \frac{\pi}{32}$$

$\rightarrow \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

Sol^n: $= \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta \, d\theta$

$$= \frac{1}{2} \beta\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma(\frac{3}{4}) \cdot \Gamma(\frac{1}{4})}{\Gamma(1)}$$

$$= \frac{1}{2} \cdot \Gamma(\tfrac{1}{4}) \cdot \Gamma(1 - \tfrac{1}{4})$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}\,\pi}{2}$$

$$= \frac{\pi}{\sqrt{2}}$$

$\rightarrow$ Prove that $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \cdot \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

Sol^n: Now, $\int_0^1 \frac{x^2\, dx}{\sqrt{1-x^4}}$

Let $x^4 = y$

$x = y^{1/4}$

$dx = \frac{1}{4} y^{\frac{1}{4}-1} dy$

when $x = 0, y = 0$

$x = 1, y = 1$

$$\therefore \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \int_0^1 \frac{y^{1/2}}{\sqrt{1-y}} \frac{1}{4} y^{\frac{1}{4}-1} dy$$

$$= \frac{1}{4} \int_0^1 y^{3/4 - 1} (1-y)^{\frac{1}{2}-1} dy$$

$$= \frac{1}{4} \beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\Gamma(\frac{3}{4}) \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4} + \frac{1}{2})}$$

$$= \frac{1}{4} \cdot \frac{\sqrt{\pi} \times \Gamma(\frac{3}{4})}{\Gamma(5/4)}$$

$$= \frac{\sqrt{\pi}}{\cancel{4}} \cdot \frac{\Gamma(\frac{3}{4})}{\cancel{\frac{1}{4}} \Gamma(\frac{1}{4})}$$

$$= \sqrt{\pi} \cdot \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}$$

Now, $\int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Let $x^2 = \tan\theta$.

$2x\,dx = \sec^2\theta\, d\theta$

$dx = \frac{1}{2\sqrt{\tan\theta}} \sec^2\theta\, d\theta$

when $x = 1, \theta = \pi/4$

$x = 0, \theta = 0$

$$\therefore \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sec\theta}\cdot\frac{1}{\sqrt{\tan\theta}}\cdot\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\sin\theta\cos\theta}}\, d\theta$$

$$= \sqrt{2}\times\sqrt{2}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\sqrt{2\sin\theta\cos\theta}}\, d\theta$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\sqrt{\sin 2\theta}}\, d\theta$$

Let $2\theta = \phi$

$$\theta = \frac{\phi}{2}$$

$$d\theta = \frac{d\phi}{2}$$

when $\theta = 0$, $\phi = 0$

$\theta = \frac{\pi}{4}$, $\phi = \frac{\pi}{2}$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sqrt{\sin\phi}}\frac{d\phi}{2}$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2}\phi, \cos^0\theta . d\phi$$

$$= \frac{1}{2\sqrt{2}}\cdot\frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}}\cdot\frac{\Gamma(\frac{1}{4})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{2})}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}}\cdot\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$$

$$\therefore \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

$$= \frac{\sqrt{\pi}\times\cancel{\Gamma(\frac{3}{4})}}{\cancel{\Gamma(1/4)}}\cdot\frac{\sqrt{\pi}}{4\sqrt{2}}\frac{\cancel{\Gamma(\frac{1}{4})}}{\cancel{\Gamma(3/4)}}$$

$$= \frac{\pi}{4\sqrt{2}}$$

Hence proved

→ Evaluate the following using Beta and gamma functions

1) $\int_0^1 x^5 . (1-x)^3 dy$

Sol: By the definition,

β- function $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$.

$$\therefore \int_0^1 x^5(1-x)^3 dx = \int_0^1 x^{6-1}(1-x)^{4-1}dx$$

$$= \beta(6,4)$$

$$= \frac{\Gamma(6)\Gamma(4)}{\Gamma(6+4)}$$

$$= \frac{5!\,3!}{9!}$$

2) $\int_0^1 \frac{1}{\sqrt{1-x^4}}dx$

Sol: Let $x^4 = y$

$$x = y^{1/4}$$

$$dx = \frac{1}{4}y^{\frac{1}{4}-1}dy.$$

$x=0$, $y=0$
$x=1$, $y=1$ } limits

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$$\therefore \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx = \int_0^1 \frac{1}{\sqrt{1-y}}\,\frac{1}{4}\,y^{\frac{1}{4}-1}\,dy$$

$$= \frac{1}{4}\int_0^1 y^{\frac{1}{4}-1}(1-y)^{\frac{1}{2}-1}\,dy$$

$$= \frac{1}{4}\,\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\cdot\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

3) $\displaystyle\int_0^\infty \frac{x^{10}-x^{18}}{(1+x)^{30}}\,dx$

Sol: By form-I,

$$\int_0^\infty \frac{x^{10}-x^{18}}{(1+x)^{30}}\,dx = \int_0^\infty \frac{x^{10}}{(1+x)^{30}}\,dx - \int_0^\infty \frac{x^{18}}{(1+x)^{30}}\,dx$$

$$= \int_0^\infty \frac{x^{11-1}}{(1+x)^{19+11}}\,dx - \int_0^\infty \frac{x^{19-1}}{(1+x)^{11+19}}\,dx$$

$$= \beta(19,11) - \beta(11,19)$$

$$= 0$$

$$(\because \beta(m,n) = \beta(n,m))$$

4) $\displaystyle\int_0^\infty \frac{x}{1+x^6}\,dx$

Sol: By form I,

Let $x^6 = y$

$x = y^{1/6}$

$dx = \frac{1}{6}\,y^{1/6-1}\,dy$

when $x=0$, $y=0$

$x=\infty$, $y=\infty$

$$\therefore \int_0^\infty \frac{y^{1/6}}{1+y}\,\frac{1}{6}\cdot y^{1/6-1}\,dy$$

$$= \frac{1}{6}\int_0^\infty \frac{y^{1/3-1}}{(1+y)^{2/3+1/3}}\,dy$$

$$= \frac{1}{6}\,\beta\left(\frac{2}{3},\frac{1}{3}\right)$$

$$= \frac{1}{6}\,\frac{\Gamma\left(\frac{2}{3}\right)\cdot\Gamma\left(\frac{1}{3}\right)}{\Gamma(1)}$$

$$= \frac{1}{6}\cdot\Gamma\left(1-\frac{1}{3}\right)\cdot\Gamma\left(\frac{1}{3}\right)$$

$$= \frac{1}{6}\cdot\frac{\pi}{\sin \pi/3}$$

$$= \frac{1}{6}\cdot\frac{2\pi}{\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}}$$

→ Prove the following.

1) $\displaystyle\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = 2\beta(10,5)$

Sol:
$$\int_0^{\infty} \frac{x^4}{(1+x)^{15}}\,dx + \int_0^{\infty} \frac{x^9}{(1+x)^{15}}\,dx$$

$$= \int_0^{\infty} \frac{x^{5-1}}{(1+x)^{10+5}}\,dx + \int_0^{\infty} \frac{x^{10-1}}{(1+x)^{5+10}}\,dx$$

$$= \beta(10,5) + \beta(5,10)$$

$$= \beta(10,5) + \beta(10,5)$$

$$= 2\beta(10,5)$$ Hence proved

2) $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^5}}\,dx = \frac{1}{5}\,\beta\left(\frac{2}{5}, \frac{1}{2}\right)$

Sol: let $x^5 = y$

$x = y^{1/5}$

$dx = \frac{1}{5} y^{\frac{1}{5}-1}\,dy.$

when $x=0$, $y=0$

$x=1$, $y=1$

$$= \int_0^1 \frac{y^{1/5}}{\sqrt{1-y}}\,\frac{1}{5}\cdot y^{\frac{1}{5}-1}\,dy$$

$$= \frac{1}{5}\int_0^1 \frac{y^{\frac{2}{5}-1}}{(1-y)^{1/2}}\cdot dy$$

$$= \frac{1}{5}\int_0^1 y^{\frac{2}{5}-1}\cdot(1-y)^{\frac{1}{2}-1}\,dy$$

By the definition,

$$\Rightarrow \int_0^1 y^{m-1}(1-y)^{n-1}\,dy = \beta(m,n)$$

$$= \frac{1}{5}\,\beta\left(\frac{2}{5}, \frac{1}{2}\right)$$

Hence proved.

3) $\displaystyle\int_0^1 x^3 (1-\sqrt{x})^5\,dx$

Sol: let $\sqrt{x} = y$

$x = y^2$

$dx = 2y\,dy$

when $x=0$, $y=0$

$x=1$, $y=1$

$$\therefore \int_0^1 y^6 (1-y)^5\, 2y\,dy$$

$$= 2\int_0^1 y^7 (1-y)^5\,dy$$

$$= 2\int_0^1 y^{8-1}(1-y)^{6-1}\,dy$$

$$= 2\,\beta(8,6)$$

$$= 2\cdot\frac{\Gamma(8)\cdot\Gamma(6)}{\Gamma(8+6)}$$

$$= 2\,\frac{7!\,5!}{13!}$$

$\rightarrow$ $\displaystyle\int_0^{\pi/2} \cos^8\theta\,d\theta$

Sol:
$$\int_0^{\pi/2} \sin^0\theta\cdot\cos^8\theta\,d\theta$$

$$= \frac{1}{2}\,\beta\left(\frac{0+1}{2}, \frac{8+1}{2}\right)$$

$$= \frac{1}{2}\,\beta\left(\frac{1}{2}, \frac{9}{2}\right)$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{9}{2}\right)}{\Gamma(5)}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{\pi}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{4!}$$

$$= \frac{35\pi}{256}$$

→ Prove that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

Sol: $\int_0^{\pi/2} \sin^{-1/2}\theta . \cos^0\theta \, d\theta$

$= \frac{1}{2} \beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)$

$= \frac{1}{2} \beta\left(\frac{1}{4}, \frac{1}{2}\right)$

$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{4}\right) . \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$

$=$

Now, $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta$

$= \int_0^{\pi/2} \sin^{1/2}\theta . \cos^0\theta \, d\theta$

$= \frac{1}{2} \beta\left(\frac{\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)$

$= \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{2}\right)$

$= \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right) . \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}$

$\therefore \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\cancel{\Gamma\left(\frac{3}{4}\right)}} . \frac{\cancel{\Gamma\left(\frac{3}{4}\right)} . \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}$

$= \frac{1}{4} \frac{\cancel{\Gamma\left(\frac{1}{4}\right)} \times \sqrt{\pi} \times \sqrt{\pi}}{\frac{1}{4} \times \cancel{\Gamma\left(\frac{1}{4}\right)}}$

$= \pi$

→ Express $\int_0^1 x^m (1-x^n)^p \, dx$ in terms of Γ-function. Hence evaluate $\int_0^1 x^5 (1-x^3)^{10} dx$

$\boxed{\text{Ans: } \frac{1}{n} \beta\left(\frac{m+1}{n}, p+1\right)}$

Sol: Let $x^n = y$

$x = y^{1/n}$

$dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$

limits when $x = 0$, $x = 1$

$y = 0$, $y = 1$

$\therefore \int_0^1 x^m (1-x^n)^p dx = \int_0^1 (y^{1/n})^m (1-y)^p \frac{1}{n} . y^{\frac{1}{n}-1} dy$

$= \frac{1}{n} \int_0^1 y^{m/n} (1-y)^p y^{1/n - 1} dy$

$= \frac{1}{n} \int_0^1 y^{\frac{m}{n}+\frac{1}{n}-1} (1-y)^p dy$

$= \frac{1}{n} \int_0^1 y^{\frac{m}{n}+\frac{1}{n}-1} (1-y)^{(p+1)-1} dy$

By the definition of beta function.

$= \frac{1}{n} . \beta\left(\frac{m+1}{n}, p+1\right)$

$= \frac{1}{n} . \frac{\Gamma\left(\frac{m+1}{n}\right) \times \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$

$= \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) . \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$

$= \frac{1}{n} . \frac{\Gamma\left(\frac{m+1}{n}\right) \times \Gamma(p+1)}{\Gamma\left(\frac{m+1+np+n}{n}\right)}$

$= \int_0^1 x^5 (1-x^3)^{10} dx$

$= \frac{1}{3} \beta\left(\frac{5+1}{3}, 10+1\right)$

$= \frac{1}{3} \beta\left(\frac{6}{3}, 11\right)$

$$= \frac{1}{3} \beta(2,11)$$

$$= \frac{1}{3} \frac{\Gamma(2)\Gamma(11)}{\Gamma(2+11)}$$

$$= \frac{1}{3} \times \frac{1! \times 10!}{\Gamma(13)}$$

$$= \frac{1}{3} \times \frac{1! \times 10!}{12!}$$

$$= \frac{1}{196}$$

VECTOR DIFFERENTIATION

→ Scalar and Vector point functions :

Consider a region in 3-D space to each point $P(x, y, z)$. Suppose, we associate a unique real number (called scalar) say "ϕ". This $\phi(x, y, z)$ is called a scalar point function defined on the region.

Similarly if to each point $P(x, y, z)$ we associate a unit vector $\bar{f}(x, y, z)$, $\bar{f}$ is called a vector point function.

For eg: 1) Take a heated solid (sphere) at each point $p(x, y, z)$ of the solid there will be temperature $T(x, y, z)$. This 'T' is a scalar point function.

2) consider a particle moving in space, at each point on its path the particle will be having a velocity 'V' which is a vector point function and the acceleration of the particle is also a vector point function.

⇒ Vector Differential operator :

It is denoted by "∇" (read as del) and is defined as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

$$\nabla = \sum i\frac{\partial}{\partial x}$$

⇒ Gradient of a scalar point function :

The Gradient of a scalar point function $\phi(x, y, z)$ is denoted by "gradϕ (or) $\nabla\phi$" and is defined as

$$\text{grad}\,\phi \text{ (or) } \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$
$$= \sum i\frac{\partial\phi}{\partial x}$$

Note: 1) The gradient of a scalar point function is a vector point function.

2) The gradient of a vector point function is not defined.

⇒ **Properties of Gradient:**

1) $\nabla\phi$ (or) grad ϕ denotes the normal vector to the level surface $\phi(x,y,z) = c$.

2) $\frac{\nabla\phi}{|\nabla\phi|}$ denotes the unit normal vector to the level surface $\phi(x,y,z) = c$

3) If 'θ' be the angle b/w two surfaces $\phi_1(x,y,z) = c_1$, $\phi_2(x,y,z) = c_2$ then $\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

⇒ **Directional Derivative: (DD):**

The directional derivative represents the rate of change of scalar point function $\phi(x,y,z)$ w.r.t. the distance at a point $p(x,y,z)$ in the direction of unit vector "$\bar{e}$" and is given by $\nabla\phi\cdot\bar{e}$.

Note: 1) The directional derivative of scalar point function $\phi(x,y,z)$ at a point $p(x,y,z)$ in the direction of any vector $\bar{a}$ is $\nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|}$

2) The directional derivative of $\phi(x,y,z)$ is maximum

in the direction of its normal, so the maximum directional derivative is $|\nabla\phi|$ i.e., the greatest rate of increase of ϕ.

3) Two surfaces $\phi_1(x,y,z) = c_1$, $\phi_2(x,y,z) = c_2$, cut orthogonally if $\boxed{\nabla\phi_1 \cdot \nabla\phi_2 = 0}$

1) Find $\nabla\phi$ if $\phi = \log(x^2+y^2+z^2)$ (scalar)

Sol:- Given, $\phi = \log(x^2+y^2+z^2)$

we have, $\text{grad}\,\phi = \nabla\phi$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Now, $\frac{\partial\phi}{\partial x} = \frac{1}{x^2+y^2+z^2} \cdot 2x$

$$\frac{\partial\phi}{\partial x} = \frac{2x}{x^2+y^2+z^2}, \quad \frac{\partial\phi}{\partial y} = \frac{2y}{x^2+y^2+z^2}$$

$$\frac{\partial\phi}{\partial z} = \frac{2z}{x^2+y^2+z^2}$$

$$\therefore \nabla\phi = i.\frac{2x}{x^2+y^2+z^2} + j.\frac{2y}{x^2+y^2+z^2} + k.\frac{2z}{x^2+y^2+z^2}$$

$$\nabla\phi = 2\left(\frac{xi+yj+zk}{x^2+y^2+z^2}\right) //$$

2) If $u = x+y+z$, $v = x^2+y^2+z^2$ and $w = xy+yz+zx$. Prove that grad u, grad v and grad w are coplanar.

Sol:- Given, $u = x+y+z$,

$v = x^2+y^2+z^2$, $w = xy+yz+zx$

we have $\text{grad}\,u = \nabla\phi$

$$\text{grad}\,u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}$$

$\therefore \text{grad}\,u = i+j+k$.

$$\text{grad}\,v = i\frac{\partial v}{\partial x} + j\frac{\partial v}{\partial y} + k\frac{\partial v}{\partial z}$$

$\therefore \text{grad}\,v = 2xi + 2yj + 2zk$.

$$\text{grad}\,w = i\frac{\partial w}{\partial x} + j\frac{\partial w}{\partial y} + k.\frac{\partial w}{\partial z}$$

$\therefore \text{grad}\,w = (y+z)i + (x+z)j + (x+y)k$

$$\therefore = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & x+z & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix}$$

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$\sim 2\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ y+z & x-y & x-z \end{vmatrix}$$

$$\sim 2(x-y)(x-z)\begin{vmatrix} 1 & 0 & 0 \\ x & -1 & -1 \\ y+z & 1 & 1 \end{vmatrix}$$

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$\sim 2(x-y)(1-z)z(-1+1)$

~ 0

$\therefore$ grad u, grad v, and grad w are coplanar.

Hence proved.

3) Find a unit vector normal to the surface $x^3+y^3+3xyz=3$ at the point $(1,2,-1)$

Sol: Let $\phi(x,y,z) = x^3+y^3+3xyz-3=0 \longrightarrow (1)$

(1) be the given surface.

$\nabla(\phi) = i.\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}$.

$\nabla(\phi) = i(3x^2+3yz)+j(3y^2+3xz)+k(3z^2+3xy)$

$\therefore \nabla(\phi)$ at $(1,2,-1)$ is

$\nabla(\phi) = [3(1)^2+3(2)(-1)]i + j[3(2)^2+3(1)(-1)]+k[(-1)^2+3(1)(2)]$

$\nabla\phi = (3-6)i+(12-3)j+(3+6)k$.

$\nabla\phi = -3i+9j+6k$

$\therefore$ The unit normal vector

$= \frac{\nabla\phi}{|\nabla\phi|}$

$= \frac{-3i+9j+6k}{\sqrt{(-3)^2+9^2+6^2}}$

$= \frac{3(-i+3j+2k)}{3\sqrt{14}}$

$= \frac{-i+3j+2k}{\sqrt{14}}$ //

4) Find a unit vector normal to the surface $xy^3z^2=4$ at the point $(1,2,-1)$.

Sol: Let $\phi(x,y,z) = xy^3z^2-4=0 \to (1)$

(1) be the given surface

$\nabla(\phi) = i.\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}$.

$\nabla(\phi) = i\,\cancel{(3x^2+3yz)}+j\,\cancel{(3y^2+3xz)}+k\,\cancel{(3z^2+3xy)}$.

$\therefore \nabla(\phi)$ at ~~(1,~~

$\nabla(\phi) = i(y^3z^2)+j(3xy^2z^2)+k(2xy^3z)$

$\therefore \nabla(\phi)$ at $(1,2,-1)$ is

$\nabla(\phi) = i(2^3(1))+j(3(1)(4)(1))+k(2(1)(8)(-1))$

$\nabla(\phi) = 8i+12j-16k$

$\therefore$ The unit normal vector

$= \frac{\nabla\phi}{|\nabla\phi|}$

$= \frac{8i+12j-16k}{\sqrt{8^2+12^2+(-16)^2}}$

$= \frac{4(2i+3j-4k)}{\sqrt{464}}$

$= \frac{4(2i+3j-4k)}{4\sqrt{29}} = 2i+3j-4k$ //

5) Find the angle between the tangent planes to the surfaces $x \log z = y^2 - 1$, $x^2 y = 2 - z$, at the point $(1,1,1)$.

Sol: $\phi_1(x,y,z) = x\log z - y^2 + 1 = 0 \rightarrow (1)$

$\phi_2(x,y,z) = x^2 y - 2 + z = 0 \rightarrow (2)$

$\nabla \phi_1 = i \frac{\partial \phi_1}{\partial x} + j \frac{\partial \phi_1}{\partial y} + k \frac{\partial \phi_1}{\partial z}$

$\nabla \phi_1 = i \log z + j(-2y) + \frac{x}{z} k$

$\nabla \phi_1 = (\log z) i - 2yj + \frac{x}{z} k$

$\nabla \phi_2 = i \frac{\partial \phi_2}{\partial x} + j \frac{\partial \phi_2}{\partial y} + k \frac{\partial \phi_2}{\partial z}$

$\nabla \phi_2 = 2xy\, i + x^2 j + k$

we know, At $(1,1,1)$

$\nabla \phi_1 = \log 1 . i - 2(1) j - 1k$
$= -2j + k$

$\nabla \phi_2 = 2i + j + k$

we know, $\cos\theta = \frac{\nabla \phi_1 . \nabla \phi_2}{|\nabla \phi_1||\nabla \phi_2|}$

$\cos\theta = \frac{(-2j+k).(2i+j+k)}{\sqrt{4+1}\sqrt{4+1+1}}$

$\cos\theta = \frac{-2+1}{\sqrt{30}}$

$\cos\theta = \frac{-1}{\sqrt{30}}$ $\quad \theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$

6) Find the values of 'a' and 'b' so that surface $5x^2 - 2yz - 9 = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at $(1,-1,2)$

Sol: Given,

$\phi_1(x,y,z) = 5x^2 - 2yz - 9 = 0 \rightarrow (1)$

$\phi_2(x,y,z) = ax^2 + by^3 - 4 = 0 \rightarrow (2)$

$\nabla \phi_1 = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$

$\nabla \phi_1 = 10xi - 2zj - 2yk$

$\nabla \phi_1$ at $(1,-1,2) = 10i - 4j + 2k$

$\nabla \phi_2 = i \frac{\partial \phi_2}{\partial x} + j \frac{\partial \phi_2}{\partial y} + k \frac{\partial \phi_2}{\partial z}$

at $(1,-1,2)$

$\nabla \phi_2 = 2axi + 3by^2 j + 0.k$
$= 2ai + 3bj$

ϕ_1 & ϕ_2 cut orthogonally

$\nabla \phi_1 . \nabla \phi_2 = 0$

i.e., $(10i - 4j + 2k).(2ai + 3bj) = 0$

$20a - 12b = 0$

given two surfaces meet at $(1,-1,2) \rightarrow (3)$

Sub $(1,-1,2)$ in equ (2)

$a - b = 4 \rightarrow (4)$

Solving (3) and (4) we get.

$a = -6, \; b = -10$

7) Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

Sol: Let $\phi(x,y,z) = xy - z^2 = 0 \rightarrow ①$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = yi + xj - 2zk$$

$\nabla\phi$ at $(4,1,2)$ is $\nabla\phi_1$

$$\nabla\phi_1 = i + 4j - 4k$$

$\nabla\phi$ at $(3,3,-3)$ is $\nabla\phi_2$

$$\nabla\phi_2 = 3i + 3j - 6k$$

If 'θ' is the angle between given two surfaces then

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$\cos\theta = \frac{(i+4j-4k)\cdot(3i+3j+6k)}{\sqrt{1+16+16}\,\sqrt{9+9+36}}$$

$$\cos\theta = \frac{3+12-24}{\sqrt{33}\sqrt{54}}$$

$$\Rightarrow \cos\theta = \frac{-9}{9\sqrt{22}} = \frac{-1}{\sqrt{22}}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{22}}\right)$$

8) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1,-2,1)$ in the direction of the vector $2i - j - 2k$.

Sol: Given, $\phi = x^2yz + 4xz^2$ and let $\bar{a} = 2i - j - 2k$, $P = (1,-2,1)$

we have $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\therefore \nabla\phi = (2xyz + 4z^2)i + x^2zj + (x^2y + 8xz)k.$$

at $(1,-2,1)$ $\nabla\phi = 0 \cdot i + j + 6k$

$$\therefore \nabla\phi = j + 6k.$$

The directional derivative of ϕ at 'p' in the direction $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (j+6k) \cdot \frac{(2i-j-2k)}{\sqrt{4+1+4}}$$

$$= \frac{-1-12}{3}$$

$$= \frac{-13}{3}$$

9) What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2 = -4$ at $(-1,2,1)$

Sol: Given, $\phi = xy^2 + yz^3$, let $P(2,-1,1)$

Let $f(x,y,z) = x\log z - y^2 + 4 = 0$

we have $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$\nabla f = \log z\, i - zy\, j + \frac{x}{z} k$

∇f at $(-1, 2, 1)$ is $\bar{a}$

$\bar{a} = -4j - k$

also $\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$

$\therefore \nabla \phi = y^2 i + (2xy + z^3) j + 3yz^2 k$

$\nabla \phi$ at $(2, -1, 1)$ is

$\nabla \phi = i - 3j - 3k$

$\therefore$ The DD of ϕ at 'p' in the direction of $\bar{a}$ is given by

$= \nabla \phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$= (i - 3j - 3k) \cdot \frac{-4j - k}{\sqrt{16+1}}$

$= \frac{12 + 3}{\sqrt{17}}$

$= \frac{15}{\sqrt{17}}$ //

10) Find the directional derivative of $\phi(x, y, z) = 5x^2y - 5y^2z + 2.5z^2x$ at the point $p(1, 1, 1)$ in the direction of the line

$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

Sol: Given, $\phi(x, y, z) = 5x^2y - 5y^2z + 2.5z^2x$

$P(1, 1, 1)$.

$\bar{a} = 2i - 2j + k$

we have $\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$

$\nabla \phi = (10xy + 2.5z^2) i + (5x^2 - 10yz) j + (-5y^2 + 5zx) k$

At $(1, 1, 1)$.

$\nabla \phi = 12.5i - 5j + 0 \cdot k$

$\nabla \phi = 12.5i - 5j$.

The DD of ϕ at p in the direction of $\bar{a}$ is given by

$= \nabla \phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$= (12.5i - 5j) \frac{(2i - 2j + k)}{\sqrt{9}}$

$= \frac{25 + 10}{3}$

$= \frac{35}{3}$ //

11) Find DD of $\phi = x^4 + y^4 + z^4$ at the point $A(1, -2, 1)$ in the direction AB where B is $(2, 6, -1)$. Also find maximum D.D. of ϕ at $(1, -2, 1)$

Sol: The position vectors of A and B with respect to the origin are $OA = i - 2j + k$ and $OB = 2i + 6j - k$.

Then $AB = OB - OA$

$AB = (2i + 6j - k) - (i - 2j + k)$

$AB = i + 8j - 2k$

$AB = \bar{a}$

we have $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$\therefore \nabla\phi = 4x^3 i + 4y^3 j + 4z^3 k$

$\nabla\phi$ at $A(1,-2,1)$ is

$\nabla\phi = 4i - 32j + 4k$.

$\therefore$ Required D.D is $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$= (4i - 32j + 4k) \cdot \frac{i + 8j - 2k}{|i + 8j - 2k|}$

$= \frac{4 - 256 - 8}{\sqrt{1 + 64 + 4}}$

$= \frac{-260}{\sqrt{69}}$

Max D.D of at $(1,-2,1)$ is

$|\nabla\phi| = \sqrt{16 + (32)^2 + 16}$

$= 4\sqrt{66}$

12) Find the values of a, b, c so that the directional derivative of $P = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum magnitude "64", in the direction parallel to the z-axis

Sol: Given, $P = axy^2 + byz + cz^2x^3$,

Let $P = (1,2,-1)$

we have, $\nabla P = i\frac{\partial P}{\partial x} + j\frac{\partial P}{\partial y} + k\frac{\partial P}{\partial z}$

$\nabla P = i(ay^2 + 3cz^2x^2) + j(2axy + bz) + k(by + 2czx^3)k$.

∇P at $(1,2,-1)$

$\nabla P = i(4a + 3c) + (4a - b)j + (2b - 2c)k$.

Since $\bar{k}$ is unit vector parallel to z-axis. $\therefore \bar{a} = \bar{k}$

$\therefore$ Required D.D $= \nabla\phi \cdot \frac{\bar{k}}{|\bar{k}|}$

$= (4a + 3c)i + (4a - b)j + (2b - 2c)k \cdot \frac{\bar{k}}{|\bar{k}|}$

$= \frac{(2b - 2c)}{1}$ ($\because k$ is a unit vector)

$= 2b - 2c$

It is given that max magnitude is 64

$\therefore 2b - 2c = 64 \rightarrow (1)$

$\because$ The vector ∇P is parallel to z-axis i.e., ⊥r to x and y-axes

i.e., $\nabla P \cdot i = 0$

$\nabla P \cdot j = 0$

i.e., $4a + 3c = 0 \rightarrow (2)$

$4a - b = 0 \rightarrow (3)$

from (1), (2) and (3) we get

$a = 6,\ b = 24,\ c = -8$

13) In what direction from $(3,1,-2)$ is the DD of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of the maximum.

Sol:- Given, $\phi = x^2y^2z^4$,

let $P(3,1,-2)$

we have,

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = i\,2xy^2z^4 + 2x^2yz^4 j + 4x^2y^2z^3 k$$

$$\nabla\phi_{(3,1,-2)} = 96i + 288j + 288k$$

$$|\nabla\phi| = 96\sqrt{19}$$

14) what is the greatest rate of increase of $u = xyz^2$ at the point $(1,0,3)$

Sol:- Given, $u = xyz^2$

let $P(1,0,3)$

$$\nabla u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}$$

$$\nabla u = i(yz^2) + j(xz^2) + k(2xyz)$$

$$\nabla u = yz^2 i + xz^2 j + 2xyz k.$$

At $(1,0,3)$ $\nabla u = 0 + 9j$

$$|\nabla u| = \sqrt{9^2} = 9$$

15) If $\bar{A} = 2x^2 i - 3yz j + xz^2 k$ and $f = 2z - x^3y$. Find (i) $\bar{A}\cdot\nabla f$ (ii) $\bar{A}\times\nabla f$ at the point $(1,-1,1)$

Sol:- Given, $\bar{A} = 2x^2 i - 3yz j + xz^2 k$ and $f = 2z - x^3y$.

we have,

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\nabla f = i(-3x^2y) + j(-x^3) + k(2)$$

$$\nabla f_{(1,-1,1)} = 3i - j + 2k$$

$\bar{A}(1,-1,1) = 2i + 3j + k$

(i) $\bar{A}\cdot\nabla f = (2i+3j+k)(3i-j+2k)$

$= 6 - 3 + 2$

$= 5$

(ii) $\bar{A}\times\nabla f = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$

$= i(6+1) - j(4-3) + k(-2-9)$

$= 7i - j - 11k$

DIVERGENCE OF VECTOR POINT FUNCTION:

The divergence of a continously differentiable vector point function '$\bar{F}$' is denoted by "$\text{div}\bar{F}$" and is defined by the equation

$$\text{div}\bar{F} = \nabla\cdot\bar{F}$$

where $\bar{F} = F_1 i + F_2 j + F_3 k$

$$\text{div}\bar{F} = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(F_1 i + F_2 j + F_3 k)$$

$$\therefore \boxed{\text{div}\bar{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}}$$

Curl of a vector point function:

The curl of a continously differentiable vector point function $\bar{F}$ is denoted by '$\text{curl}\bar{F}$' and is defined by the equation

$\text{curl}\bar{F} = \nabla\times\bar{F}$, where

$\bar{F} = F_1 i + F_2 j + F_3 k$.

$$\text{curl}\bar{F} = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(F_1 i + F_2 j + F_3 k)$$

$$\text{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= i\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

Note: 1) The divergence of vector point function (V.P.F) is a Scalar point function (S.P.F)

2) The vector point function $\bar{F}$ is said to be solenoidal if $\text{div}\bar{F} = 0$.

3) The curl of a vector point function is a vector point function.

4) If $\text{curl}\bar{F} = 0$, then $\bar{F}$ is called irrotational (conservative) vector point function.

⇒ Laplacian Operator:

The scalar differential operator $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is called Laplacian operator. If 'ϕ' is any scalar point function then

$$\text{div}(\text{grad}\,\phi) = \nabla(\nabla\phi)$$

$$\text{div}(\text{grad}\,\phi) = \nabla^2\phi$$

$$\text{div}(\text{grad}\,\phi) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi$$

$$\boxed{\text{div}(\text{grad}\,\phi) = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}}$$

→ $\nabla^2\phi = 0$ is called Laplacian Equation.

1) Evaluate $\operatorname{div}\bar{F}$ and $\operatorname{curl}\bar{F}$ at the point $(1,2,3)$ where $\bar{F} = x^2yz\,i + xy^2z\,j + xyz^2\,k$

Sol: - Given, $\bar{F} = x^2yz\,i + xy^2z\,j + xyz^2\,k \rightarrow ①$

Let $P = (1,2,3)$

we have, $\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Equ ① is of the form

$\bar{F} = F_1 i + F_2 j + F_3 k$ where

$F_1 = x^2yz$, $F_2 = xy^2z$, $F_3 = xyz^2$

$\therefore \operatorname{div}\bar{F} = 2xyz + 2xyz + 2xyz$

$\operatorname{div}\bar{F} = 6xyz$

At $P(1,2,3)$ $\operatorname{div}\bar{F} = 6(1)(2)(3)$

$\therefore \boxed{\operatorname{div}\bar{F} = 36}$

we have, $\operatorname{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$\operatorname{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$\operatorname{curl}\bar{F} = i(xz^2 - xy^2) - j(yz^2 - x^2y) + k(y^2z - x^2z)$

$\operatorname{curl}\bar{F}$ at $(1,2,3)$ is

$= 5i + 16j + 9k.$

2) Find $\operatorname{div}\bar{F}$ and $\operatorname{curl}\bar{F}$ where $\bar{F} = \operatorname{grad}(x^3y + y^3z + z^3x - x^2y^2z^2)$

Sol: - Given,

$\bar{F} = \operatorname{grad}(x^3y + y^3z + z^3x - x^2y^2z^2)$

Let $\phi = x^3y + y^3z + z^3x - x^2y^2z^2$

$\operatorname{grad}\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\operatorname{grad}\phi = (3x^2y + z^3 - 2xy^2z^2)i + j(x^3 + 3y^2z - 2x^2yz^2) + k(y^3 + 3xz^2 - 2x^2y^2z)$

$\therefore \bar{F} = \operatorname{grad}\phi = (3x^2y + z^3 - 2xy^2z^2)i + (x^3 + 3y^2z - 2x^2yz^2)j + (y^3 + 3xz^2 - 2x^2y^2z)k.$

we have

$\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$\therefore \operatorname{div}\bar{F} = (6xy - 2y^2z^2) + (6yz - 2x^2z^2) + (6xz - 2x^2y^2)$

Now,

$$\operatorname{Curl}\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y + z^3 - 2xy^2z^2 & F_2 & F_3 \end{vmatrix}$$

$\operatorname{Curl}\bar{F} = i\,(3y^2 - 4x^2yz) -$

3) If $u = x^2+y^2+z^2$, $\bar{V} = xi+yj+zk$. Show that $\operatorname{div}(u\bar{V}) = 5u$.

Sol: Given, $u = x^2+y^2+z^2$

$\bar{V} = xi+yj+zk$.

$u\bar{V} = (x^2+y^2+z^2)(xi+yj+zk)$

$u\bar{V} = (x^3+xy^2+xz^2)i+(x^2y+y^3+yz^2)j + (x^2z+y^2z+z^3)k$

$= \bar{F}$, say.

we have, $\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$

$= \frac{\partial}{\partial x}(x^3+xy^2+xz^2)+\frac{\partial}{\partial y}(x^2y+y^3+yz^2) + \frac{\partial}{\partial z}(x^2z+y^2z+z^3)$

$= 3x^2+y^2+z^2+x^2+3y^2+z^2+x^2+y^2+3z^2$

$= 5x^2+5y^2+5z^2$

$= 5(x^2+y^2+z^2)$

$= 5u$

4) If $\bar{F} = (x+y+1)i+j-(x+y)k$. Show that $\bar{F}.\operatorname{curl}\bar{F} = 0$.

Sol: Given,

$\bar{F} = (x+y+1)i+j-(x+y)k$.

we have,

$$\operatorname{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\operatorname{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$\operatorname{curl}\bar{F} = i(-1-0)-j(-1-0)+k(0-1)$

$\therefore \operatorname{curl}\bar{F} = -i+j-k$

Now, $\bar{F}.\operatorname{curl}\bar{F} = [(x+y+1)i+j + (x+y)k].[-i+j-k]$

$\bar{F}.\operatorname{curl}\bar{F} = -(x+y+1)+1+(x+y)$

$\bar{F}.\operatorname{curl}\bar{F} = -x-y-1+1+x+y$

$\therefore \boxed{\bar{F}.\operatorname{curl}\bar{F} = 0}$ Hence proved

5) Calculate curl (grad f), given $f(x,y,z) = x^2+y^2-z$.

Sol: Given,

$f = x^2+y^2-z$.

$\operatorname{grad} f = i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}$.

$\operatorname{grad}\phi = 2xi+2yj-k$.

$$\operatorname{curl}(\operatorname{grad} f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\operatorname{curl}(\operatorname{grad} f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & -1 \end{vmatrix}$$

$\operatorname{curl}(\operatorname{grad} f) = i(0-0)-j(0-0)+k(0-0)$

$\therefore \operatorname{curl}(\operatorname{grad} f) = 0$

6) Calculate curl(curl $\bar{A}$), given $\bar{A} = x^2y\,i + y^2z\,j + z^2x\,k$

Sol: Given, $\bar{A} = x^2y\,i + y^2z\,j + z^2x\,k$

$$\text{curl}\,\bar{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$$\text{curl}\,\bar{A} = i(0 - y^2) - j(z^2 - 0) + k(0 - x^2)$$

$$\text{curl}\,\bar{A} = -y^2 i - z^2 j - x^2 k$$

Let curl $\bar{A} = \bar{F}$

$$\therefore \text{curl}(\bar{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & -z^2 & -x^2 \end{vmatrix}$$

$$= i(0 + 2z) - j(-2x - 0) + k(0 + 2y)$$

$$= +2z\,i + 2x\,j + 2y\,k$$

$$\text{curl}\,\bar{F} = 2(z\,i + x\,j + y\,k)$$

7) If $u = x^2yz$, $v = xy - 3z^2$, find (i) $\nabla(\nabla u \cdot \nabla v)$ (ii) $\nabla \cdot (\nabla u \times \nabla v)$

Sol: Given, $u = x^2yz$, $v = xy - 3z^2$

(i) $\nabla u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}$

$$\nabla u = 2xyz\,i + x^2z\,j + x^2y\,k$$

$$\nabla v = i\frac{\partial v}{\partial x} + j\frac{\partial v}{\partial y} + k\frac{\partial v}{\partial z}$$

$$\nabla v = y\,i + x\,j - 6z\,k$$

$$\nabla u \cdot \nabla v = 2xy^2z + x^3z - 6x^2yz = F$$

$$\nabla(\nabla u \cdot \nabla v) = i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}$$

$$\nabla(F) = (2y^2z + 3x^2)\,i + (xyz - 6x^2z)\,j + (2xy^2 + x^3 - 6x^2y)\,k$$

(ii) $$\nabla u \times \nabla v = \begin{vmatrix} i & j & k \\ 2xyz & x^2z & x^2y \\ y & x & -6z \end{vmatrix}$$

$$\nabla u \times \nabla v = i(-6x^2z^2 - x^3y) - j(-12xyz^2 - x^2y^2) + k(2x^2yz - x^2yz)$$

Let $\nabla u \times \nabla v = \bar{G}$, say

$$\nabla(\bar{G}) = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial y}$$

$$= (-12xz^2 - 3x^2y) + (12xz^2 + 2x^2y) + (2x^2y - x^2y)$$

$$= 0$$

$$\therefore \nabla \cdot (\nabla u \times \nabla v) = 0$$

8) Determine the constant such that $\bar{A} = (bx + 4y^2z)\,i + (x^2 \sin z - 3y)\,j - (e^x + 4\cos x^2y)\,k$. If $\bar{A}$ is solenoidal

Sol: Given, $\bar{A} = (bx + 4y^2z)\,i + (x^2 \sin z - 3y)\,j - (e^x + 4\cos x^2y)\,k$.

By given, $\bar{A}$ is solenoidal then $\text{div}\,\bar{A} = 0$.

$\operatorname{div} \bar{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$

$\therefore \operatorname{div}\bar{A} = b - 3 + 0$

$\therefore b - 3 = 0$

$b = 3$

9) Find the value of 'a', $\bar{F} = (ax^2y + yz)i + (xy^2 - xz^2)j + (2xyz - 2x^2y^2)k$. If $\bar{F}$ has zero divergence.

Sol:- Given, $\bar{F} = (ax^2y + yz)i + (xy^2 - xz^2)j + (2xyz - 2x^2y^2)k$

By given, F has 0 divergence.

i.e., $\operatorname{div}\bar{F} = 0$

$\operatorname{div}F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$

$\operatorname{div}\bar{F} = \cancel{(y^2 - z^2)}\ 2axy + 2xy + 2xy = 0$

$2axy = -4xy$

$\therefore \boxed{a = -2}$

10) If $\bar{r} = xi + yj + zk$, show that (i) $\nabla \bar{r} = 3$, (ii) $\nabla \times \bar{r} = 0$

Sol:- Given, $\bar{r} = xi + yj + zk$.

we have, $\nabla \bar{r} = \frac{\partial r_1}{\partial x} + \frac{\partial r_2}{\partial y} + \frac{\partial r_3}{\partial z}$

$\nabla \bar{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$

$\nabla \bar{r} = 1 + 1 + 1$

$\nabla \bar{r} = 3$

Also we have,

$$\nabla \times \bar{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$\nabla \times \bar{r} = i(0-0) - j(0-0) + k(0-0)$

$\nabla \times \bar{r} = 0$

11) Find the constants A, B, C Such that the vector ~~$\bar{A} =$~~ $\bar{A} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is irrotational.

Sol:- Given that $\bar{A}$ is irrotational, then $\operatorname{curl}\bar{A} = 0$

i.e.,

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$= 0.i + 0.j + 0.k$.

$\Rightarrow i(c+1) - j(4-a) + k(b-2) = 0.i + 0.j + 0.k$.

$\Rightarrow i(c+1) + j(a-4) + k(b-2) = 0.i + 0.j + 0.k$

comparing i, j, k co-efficients.

$c + 1 = 0, \quad a - 4 = 0, \quad b - 2 = 0$

$a = 4, \ b = 2, \ c = -1$

12) calculate $\nabla^2 f$ when $f = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$ at the point $(1,1,0)$.

Sol: Given, $f = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$

we have, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$\therefore \nabla^2 f = (6z - 0 + 24xy) + (0 - 2z^3) + (-6y^2z)$

$\nabla^2 f$ at $(1,1,0) = (0 + 24(1)(1)) + (0 - 0) + (-0)$

$= 24$

$\therefore \boxed{\nabla^2 f = 24}$

13) Find directional derivative of $\nabla \cdot (\nabla \phi)$ at the point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $\phi = 2x^3y^2z^4$

Sol: Given, $\phi = 2x^3y^2z^4$.

Let $g = xy^2z - 3x - z^2 = 0$

$\nabla\phi = i \cdot \frac{\partial \phi}{\partial x} + j \cdot \frac{\partial \phi}{\partial y} + k \cdot \frac{\partial \phi}{\partial y}$

$\nabla\phi = 6x^2y^2z^4 i + 4x^3yz^4 j + 8x^3y^2z^3 k$

$(\nabla\phi) = \nabla(F) = \frac{\partial F1}{\partial x} + \frac{\partial F2}{\partial y} + \frac{\partial F3}{\partial z}$

$\nabla(\nabla\phi) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

Let $\nabla \cdot (\nabla\phi) = f1$, say

$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$.

$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$.

$\nabla f = (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)i + (24xyz^4 + 48x^3yz^2)j + (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)k$

∇f at $(1,-2,1) = (12(-2)^2(1)^4 + 12(1)^2(1)^4 + 72(1)^2(-2)^2(1))i + (24(1)(-2)(1)^4 + 48(1)(-2)(1)^2)j + (48(1)(-2)^2(1)^3 + 16(1)^3(1)^3 + 48(1)^3(-2)^2(1))k$

$\therefore \nabla f$ at $(1,-2,1)$

$= 348i - 144j + 400k$

also given, $g = xy^2z - 3x - z^2 = 0$

$\nabla g = i \frac{\partial g}{\partial x} + j \frac{\partial g}{\partial y} + k \frac{\partial g}{\partial z}$

$\therefore \nabla g = (y^2z - 3)i + (2xyz)j + (xy^2 - 2z)k$.

At $(1,-2,1)$

$\nabla g = i - 4j + 2k = \bar{a}$, say

$\therefore$ Required D.D $= \bar{f} \cdot \frac{\bar{a}}{|\bar{a}|}$

$= (348i - 144j + 400k) \cdot \frac{i - 4j + 2k}{\sqrt{1^2 + 16 + 4}}$

$= \frac{348 + 576 + 800}{\sqrt{21}}$

$= \frac{1724}{\sqrt{21}}$

14) If $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$ then (i) prove that $\nabla r^n = n r^{n-2} \bar{r}$ (ii) $\mathrm{div}(r^n \bar{r}) = (n+3)r^n$.

Hence show that $\frac{\bar{r}}{r^3}$ is solenoidal. (iii) show that $r^n\bar{r}$ is irrotational (iv) Show that $\mathrm{div}(\mathrm{grad}\, r^n) = \nabla^2 r^n = n(n+1) r^{n-2}$

Sol: Given, $\bar{r} = xi + yj + zk$ and

$r = |\bar{r}|$ i.e., $r = |xi + yj + zk|$

$\therefore r = \sqrt{x^2+y^2+z^2}$

Differentiate 'r' partially w.r.t. x, y, and z, we get

$$\therefore \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$ similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

(i) $\nabla r^n = \sum i . \frac{\partial r^n}{\partial x}$

$$= \sum i . n . r^{n-1} . \frac{\partial r}{\partial x}$$

$$= \sum i . n . r^{n-1} . \frac{x}{r}$$

$$= \sum i n . r^{n-2} . x$$

$$= n . r^{n-2} \sum x i$$

$$= n . r^{n-2} (xi + yj + zk)$$

$$\therefore \boxed{\nabla r^n = n r^{n-2} \bar{r}}$$

(ii) $\mathrm{div}(r^n\bar{r}) =$

Consider $r^n\bar{r} = r^n(xi + yj + zk)$

$r^n\bar{r} = r^n x i + r^n y j + r^n z k$

$\mathrm{div}(r^n\bar{r}) = \sum \cancel{i} . \frac{\partial}{\partial x}(x r^n)$ $\left(\because \frac{d}{dx}(u,v)\right)$

$$\mathrm{div}(r^n\bar{r}) = \sum \left(r^n + x . n . r^{n-1} . \frac{\partial r}{\partial x}\right)$$

$$= \sum \left(r^n + x . n r^{n-1} \frac{x}{r}\right)$$

$$= 3r^n + \sum n r^{n-2} x^2$$

$$= 3r^n + n . r^{n-2} \sum x^2$$

$$= 3r^n + n . r^{n-2}(x^2+y^2+z^2)$$

$$= 3r^n + n r^{n-2} . r^2$$

$$= 3r^n + n r^n$$

$$= r^n(3+n)$$

$\therefore \mathrm{div}(r^n\bar{r}) = (n+3) r^n$

if $n = -3$, $\mathrm{div}(r^{-3}\bar{r}) = 0$

$$\mathrm{div}\left(\frac{\bar{r}}{r^3}\right) = 0$$

$\therefore \frac{\bar{r}}{r^3}$ is solenoidal

(iii) $r^n . \bar{r} = r^n(xi + yj + zk)$

$r^n . \bar{r} = x r^n i + y r^n j + z r^n k = \bar{F}$, say.

$$\mathrm{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x r^n & y r^n & z r^n \end{vmatrix}$$

$$\mathrm{curl}\,\bar{F} = \sum i \left(\frac{\partial}{\partial y}(z r^n) - \frac{\partial}{\partial z}(y r^n)\right)$$

$$\mathrm{curl}\,\bar{F} = \sum i \left(z . n . r^{n-1} \frac{\partial r}{\partial y} - y . n . r^{n-1} \frac{\partial r}{\partial z}\right)$$

$$\mathrm{curl}\,\bar{F} = \sum i \left(z . n . r^{n-1} . \frac{y}{r} - y . n . r^{n-1} \frac{z}{r}\right)$$

$$\mathrm{curl}\,\bar{F} = \sum i \left(y z n . r^{n-2} - y z n . r^{n-2}\right)$$

$$\mathrm{curl}\,\bar{F} = 0$$

$\therefore \mathrm{curl}\,(r^n\bar{r}) = 0$

$\therefore r^n\bar{r}$ is irrotational

(iv) $\mathrm{grad}\, r^n = i \frac{\partial r^n}{\partial x} + j \frac{\partial r^n}{\partial y} + k \frac{\partial r^n}{\partial z}$

$$\mathrm{grad}\, r^n = \sum i \frac{\partial r^n}{\partial x}$$

$$\mathrm{grad}\, r^n = \sum i . n . r^{n-1} \frac{\partial r}{\partial x}$$

$\text{grad } r^n = \sum i\, n\, r^{n-1} \cdot \frac{x}{r}$

$\text{grad } r^n = n \sum r^{n-2} \cdot x \cdot i$

$\therefore \text{div}(\text{grad } r^n) = n \sum \frac{\partial}{\partial x}(r^{n-2} x)$ $\quad \frac{d}{dx}(uv)$

$= n \sum \left[r^{n-2}(1) + x(n-2) \cdot r^{n-3} \cdot \frac{\partial r}{\partial x} \right]$

$= n \sum \left[r^{n-2} + x(n-2)\, r^{n-3} \frac{x}{r} \right]$

$= n \left[3 r^{n-2} + (n-2)\, r^{n-4} \cdot \sum x^2 \right]$

$= n \left(3 r^{n-2} + (n-2) \cdot r^{n-4} \cdot r^2 \right)$

$= n \left(3 r^{n-2} + (n-2) \cdot r^{n-2} \right)$

$= n\, r^{n-2} \left(3 + (n-2) \right)$

$= n \cdot r^{n-2} (n+1)$

$= n(n+1) \cdot r^{n-2}$

$\therefore \text{div}(\text{grad } r^n) = \nabla^2 r^n = n(n+1) r^{n-2}$

Hence proved.

⇒ Scalar Potential Function

If $\bar{F}$ is irrotational then $\exists$ a function $\phi(x,y,z)$ such that $\bar{F} = \nabla \phi$ (or) $\bar{F} = \text{grad } \phi$ then "ϕ" is called "Scalar Potential Function" of $\bar{F}$.

If $\bar{F} = F_1 i + F_2 j + F_3 k$

$\bar{F} = \nabla \phi$

$F_1 i + F_2 j + F_3 k = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$

$\therefore F_1 = \frac{\partial \phi}{\partial x}, \quad F_2 = \frac{\partial \phi}{\partial y}, \quad F_3 = \frac{\partial \phi}{\partial z}$

we know that, $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$

1) A vector field is given by
$\bar{F} = (x^2-y^2+x)i - (2xy+y)j + 0.k$. Show that the field is irrotational and find its scalar potential.

Sol:- Given, $\bar{F} = (x^2-y^2+x)i - (2xy+y)j + 0.k$.

$$\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2+x & -2xy-y & 0 \end{vmatrix}$$

$= i(0-0) - j(0-0) + k(-2y+2y)$

$= 0$

$\therefore$ curl $\bar{F} = 0$

$\therefore$ F is irrotational.

Then $\exists$ 'ϕ' such that

$\bar{F} = \nabla\phi$

Here $(x^2-y^2+x)i - (2xy+y)j + 0.k$

$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k.\frac{\partial\phi}{\partial z}$.

By comparing i, j, k co-efficients.

$x^2-y^2+x = \frac{\partial\phi}{\partial x}$, $-2xy-y = \frac{\partial\phi}{\partial y}$

and $\frac{\partial\phi}{\partial z} = 0$

we know that,

$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$d\phi = (x^2-y^2+x)dx + (-2xy-y)dy + (0)dz$

$d\phi = x^2dx - y^2dx + xdx - 2xydy - ydy$

$d\phi = x^2dx + xdx - ydy - y^2dx - 2xydy$.

$d\phi = x^2dx + xdx - ydy - (y^2dx + 2xydy)$

$d\phi = x^2dx + xdx - ydy - (\overset{d(uv)}{d(y^2x)})$

Integrating on both sides.

$\int d\phi = \int x^2dx + \int xdx - \int ydy - \int d(y^2x)$

$$\boxed{\phi = \frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - y^2x}$$

LINE INTEGRALS

→ Any integral which is to be evaluated along a curve is called a line integral.

→ The line integral of a continuous vector point function $\vec{F}$ along the curve C is denoted by $\int_C \vec{F}\cdot d\vec{r}$

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ so that $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

Note: If C is a closed curve, then the integral sign $\int_C$ is replaced by $\oint_C$.

Applications of line integral:

(i) Work done by a force: If $\vec{F}$ represents the force acting on a particle moving along an arc AB then the total work done by $\vec{F}$ during displacement from A to B is given by $\int_A^B \vec{F}\cdot d\vec{r}$

(ii) Circulation: If $\vec{F}$ represents the velocity of a fluid particle and C is a closed curve then $\oint_C \vec{F}\cdot d\vec{r}$ is called the circulation of $\vec{F}$ round the curve C.

Note: If $\oint_C \vec{F}\cdot d\vec{r} = 0$ then $\vec{F}$ is said to be irrotational.

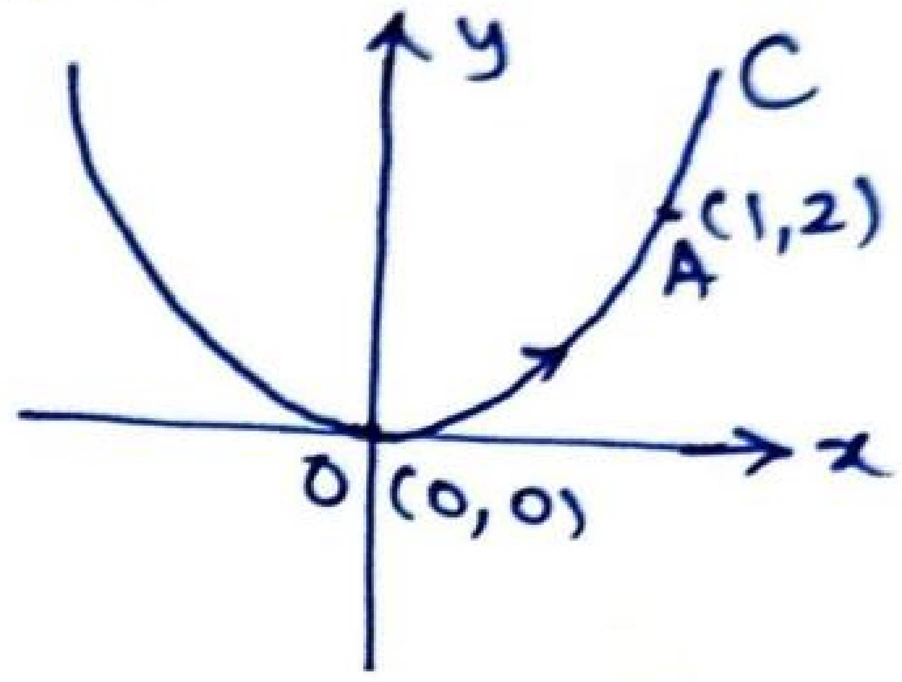
Example 1. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve $y = 2x^2$ in the xy-plane from (0,0) to (1,2).

Solution: Given $\vec{F} = 3xy\hat{i} - y^2\hat{j}$

Here $F_1 = 3xy$; $F_2 = -y^2$; $F_3 = 0$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy \quad (\because \text{In } xy\text{-plane}, \ z=0 \Rightarrow dz=0)$$

$$= \int_C 3xy\,dx - y^2\,dy$$



Along C, $y = 2x^2$ so that $dy = 4x\,dx$

and $0 \le x \le 1$

$$= \int_0^1 3x(2x^2)\,dx - (2x^2)^2\cdot(4x)\,dx$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= \left(6\cdot\frac{x^4}{4} - 16\cdot\frac{x^6}{6}\right)_0^1 = \frac{6}{4} - \frac{16}{6} = -\frac{7}{6}$$

Hence $\int_C \vec{F}\cdot d\vec{r} = -\frac{7}{6}$

Example 2. If $\vec{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int \vec{A}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path $x=t$, $y=t^2$, $z=t^3$.

Solution: Here $A_1 = 3x^2+6y$; $A_2 = -14yz$; $A_3 = 20xz^2$

$$\therefore \int_C \vec{A}\cdot d\vec{r} = \int_C (A_1\,dx + A_2\,dy + A_3\,dz)$$

$$= \int_C \left[(3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz\right]$$

Along C: $x=t$, $y=t^2$, $z=t^3$ so that $dx=dt$, $dy=2t\,dt$

and $dz = 3t^2\,dt$, $0 \le t \le 1$

$\boxed{\begin{array}{l}(0,0,0) \Rightarrow 0=t,\ 0=t^2,\ 0=t^3 \\ \qquad \Rightarrow t=0 \\ (1,1,1) \Rightarrow 1=t,\ 1=t^2,\ 1=t^3 \Rightarrow t=1\end{array}}$

$$= \int_{t=0}^1 \left[(3t^2+6t^2)\,dt - 14(t^2)(t^3)(2t)\,dt + 20(t)(t^3)^2(3t^2)\,dt\right]$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

i.e., $$\int_C \vec{A}\cdot d\vec{r} = \left(\cancel{9}^{3}\cdot\frac{t^3}{\cancel{3}} - \cancel{28}^{4}\cdot\frac{t^7}{\cancel{7}} + \cancel{60}^{6}\cdot\frac{t^{10}}{\cancel{10}}\right)_0^1 = 3 - 4 + 6 = 5$$

Example 3. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$, along

(a) the straight line from $(0,0,0)$ to $(2,1,3)$

(b) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x=0$ to $x=2$

Solution: Here $F_1 = 3x^2$; $F_2 = (2xz-y)$; $F_3 = z$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C [F_1 dx + F_2 dy + F_3 dz] = \int_C [3x^2 dx + (2xz-y)dy + z\,dz]$$

(a) Equations of the straight line joining $A(0,0,0)$ and $B(2,1,3)$ is $\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$, t is the parameter.

i.e., $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$

$\Rightarrow x = 2t,\ y = t,\ z = 3t$

$$\therefore \text{Work done} = \int_A^B \vec{F}\cdot d\vec{r}$$

$$= \int_A^B [3x^2 dx + (2xz-y)dy + z\,dz]$$

Along the straight line AB: $x = 2t, y = t, z = 3t$ so that $dx = 2dt$, $dy = dt$ and $dz = 3dt$. where $0 \le t \le 1$

$A(0,0,0) \Rightarrow 0 = 2t,\ 0 = t$ and $0 = 3t \therefore t = 0$
$B(2,1,3) \Rightarrow 2 = 2t,\ 1 = t$ and $3 = 3t \therefore t = 1$

$$= \int_{t=0}^{1} 3(2t)^2(2\,dt) + (2(2t)(3t) - t)dt + (3t)(3\,dt)$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\,dt$$

$$= \int_0^1 (36t^2 + 8t)\,dt = \left(\overset{12}{\cancel{36}}\cdot\frac{t^3}{\cancel{3}} + \overset{4}{\cancel{8}}\cdot\frac{t^2}{\cancel{2}}\right)_0^1 = 12 + 4 = 16$$

(b) Let $x = t$. Then $t^2 = 4y$ i.e., $y = \frac{t^2}{4}$ and $3t^3 = 8z \Rightarrow z = \frac{3}{8}t^3$

$$\therefore \text{Work done} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C [3x^2 dx + (2xz-y)dy + z\,dz]$$

Along C: $x=t$, $y=\frac{t^2}{4}$, $z=\frac{3}{8}t^3$ so that $dx=dt$, $dy=\frac{t}{2}dt$ 4

and $dz=\frac{9}{8}t^2dt$, where $0\le t\le 2$ $(\because 0\le x\le 2)$

$$= \int_{t=0}^{2}\left[3t^2dt + \left(\cancel{2}(t)\left(\frac{3}{\cancel{8}_4}t^3\right)-\frac{t^2}{4}\right)\left(\frac{t}{2}\right)dt + \frac{3}{8}t^3\left(\frac{9}{8}t^2\right)dt\right]$$

$$= \int_0^2\left(3t^2+\frac{3}{8}t^5-\frac{t^3}{8}+\frac{27}{64}t^5\right)dt$$

$$= \int_0^2\left(3t^2-\frac{t^3}{8}+\frac{51}{64}t^5\right)dt$$

$$= \left(\cancel{3}\cdot\frac{t^3}{\cancel{3}}-\frac{1}{8}\cdot\frac{t^4}{4}+\frac{51}{64}\cdot\frac{t^6}{6}\right)_0^2$$

$$= (2)^3-\frac{1}{32}(2)^4+\frac{51}{\cancel{64}}\cdot\frac{\cancel{(2)^6}}{6}$$

$\therefore$ Work done $= 8-\frac{16}{32}+\frac{51}{6} = 16$

Example 4. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=[2z, x, -y]$ and C is $\vec{r}=[\cos t, \sin t, 2t]$ from $(1,0,0)$ to $(1,0,4\pi)$

Solution: Given $\vec{F}=[2z,x,-y]=2z\,\hat{i}+x\hat{j}-y\,\hat{k}$

and $\vec{r}=[\cos t, \sin t, 2t] \Rightarrow x=\cos t$, $y=\sin t$ and $z=2t$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C F_1dx+F_2dy+F_3dz$$

$$= \int_C 2z\,dx + x\,dy - y\,dz$$

Along C: $x=\cos t$, $y=\sin t$, $z=2t$ so that $dx=-\sin t\,dt$, $dy=\cos t\,dt$, $dz=2dt$; where $0\le t\le 2\pi$

$$\boxed{\begin{array}{l}(1,0,0)\Rightarrow 1=\cos t, 0=\sin t, 0=2t \quad \therefore t=0\\ (1,0,4\pi)\Rightarrow 1=\cos t, 0=\sin t, 4\pi=2t \quad \therefore t=2\pi\end{array}}$$

$$= \int_{t=0}^{2\pi}\left[2(2t)(-\sin t)dt + \cos t(\cos t)\,dt - \sin t(2)\,dt\right]$$

$$= \int_0^{2\pi}\left(-4t\sin t+\cos^2 t-2\sin t\right)dt$$

$$= -4\int_0^{2\pi} t\sin t\,dt + \frac{1}{2}\int_0^{2\pi} 2\cos^2 t\,dt - 2\int_0^{2\pi}\sin t\,dt$$

$$= -4\left[t\cdot(-\cos t) - \int 1\cdot(-\cos t)\,dt\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi}(1+\cos 2t)\,dt - 2(-\cos t)\Big|_0^{2\pi}$$

$$= -4\left[-t\cos t + \sin t\right]_0^{2\pi} + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} + 2(\cos 2\pi - \cos 0)$$

$$= -4\left[(-2\pi\cos 2\pi + \sin 2\pi) - 0\right] + \frac{1}{2}\left[\left(2\pi + \frac{\sin 4\pi}{2}\right) - 0\right] + 2(1-1)$$

$$= -4(-2\pi + 0) + \frac{1}{2}(2\pi + 0) + 2(0)$$

$$= 8\pi + \pi$$

$$= 9\pi$$

Hence $\int_C \vec{F}\cdot d\vec{r} = 9\pi$

Example 5. If $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$, evaluate $\int_C \vec{F}\times d\vec{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \pi/2$

Solution: Given $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$

Since $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Now
$$\vec{F}\times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix} = \hat{i}(-z\,dz - x\,dy) - \hat{j}(2y\,dz - x\,dx) + \hat{k}(2y\,dy + z\,dx)$$

Along C: $x = \cos t$, $y = \sin t$, $z = 2\cos t$ so that $dx = -\sin t\,dt$, $dy = \cos t\,dt$, $dz = -2\sin t\,dt$ and $0 \le t \le \frac{\pi}{2}$

$$\therefore \vec{F}\times d\vec{r} = \hat{i}\left[(-2\cos t)(-2\sin t)\,dt - (\cos t)(\cos t)\,dt\right]$$
$$- \hat{j}\left[(2\sin t)(-2\sin t)\,dt - (\cos t)(-\sin t)\,dt\right] + \hat{k}\left[(2\sin t)(\cancel{\cos t})\,dt + (2\cancel{\cos t})(-\sin t)\,dt\right]$$

$$= \hat{i}(4\sin t\cos t - \cos^2 t)\,dt + \hat{j}(4\sin^2 t - \sin t\cos t)\,dt$$

$$\therefore \int_C \vec{F}\times d\vec{r} = \int_{t=0}^{\pi/2}\left[\hat{i}(4\sin t\cos t - \cos^2 t)\,dt + \hat{j}(4\sin^2 t - \sin t\cos t)\,dt\right]$$

$$= \hat{i}\int_0^{\pi/2}\left[2(2\sin t\cos t) - \frac{1}{2}(2\cos^2 t)\right]dt + \hat{j}\int_0^{\pi/2}\left[2(2\sin^2 t) - \frac{1}{2}(2\sin t\cos t)\right]dt$$

$$= \hat{i}\int_0^{\pi/2}\left[2\sin 2t - \frac{1}{2}(1+\cos 2t)\right]dt + \hat{j}\int_0^{\pi/2}\left[2(1-\cos 2t) - \frac{1}{2}\sin 2t\right]dt$$

$$= \hat{i}\left[\cancel{2}\left(-\frac{\cos 2t}{\cancel{2}}\right)-\frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\right]_0^{\pi/2}+\hat{j}\left[2\left(t-\frac{\sin 2t}{2}\right)-\frac{1}{2}\left(-\frac{\cos 2t}{2}\right)\right]_0^{\pi/2}$$

$$= \hat{i}\left[-\cos \cancel{2}\cdot\frac{\pi}{\cancel{2}}-\frac{1}{2}\left(\frac{\pi}{2}+\frac{\sin \cancel{2}\cdot\pi/\cancel{2}}{2}\right)+\cos 0\right]+\hat{j}\left[2\left(\frac{\pi}{2}-\frac{\sin \cancel{2}\cdot\pi/\cancel{2}}{2}\right)+\frac{1}{4}\cos \cancel{2}\frac{\pi}{\cancel{2}}-\frac{1}{4}\cos 0\right]$$

$$= \hat{i}\left(-\cos\pi-\frac{\pi}{4}+1\right)+\hat{j}\left(\pi+\frac{1}{4}(-1)-\frac{1}{4}\right)$$

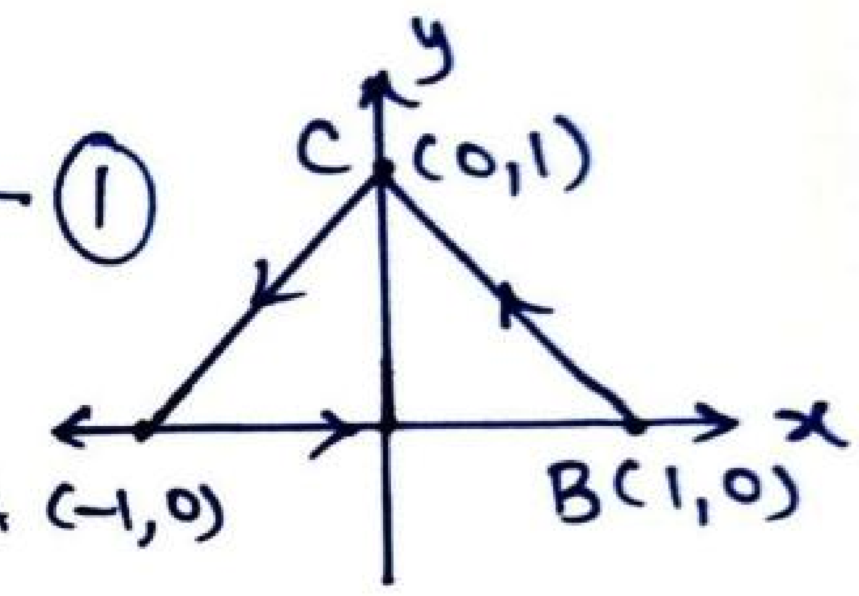
$$= \hat{i}\left(1-\frac{\pi}{4}+1\right)+\hat{j}\left(\pi-\frac{1}{2}\right)$$

$$= \left(2-\frac{\pi}{4}\right)\hat{i}+\left(\pi-\frac{1}{2}\right)\hat{j}$$

Example 6. Compute the line integral $\oint_C (y^2dx - x^2dy)$ about the triangle whose vestices are $(1,0)$, $(0,1)$ and $(-1,0)$.

Solution:

we have $\oint_C (y^2dx - x^2dy) = \int_{AB} + \int_{BC} + \int_{CA}$ —— (1)



Along AB: $y=0$ so that $dy=0$ and $-1 \le x \le 1$

$$\therefore \int_{AB}(y^2dx - x^2dy) = \int_{AB}(0) = 0$$

Along BC: Equation of line BC is $y-0 = \frac{1-0}{0-1}(x-1)$ i.e., $y = 1-x$ so that $dy = -dx$ and $1 \le x \le 0$.

$$\therefore \int_{BC}(y^2dx - x^2dy) = \int_{x=1}^{0}(1-x)^2dx - x^2(-dx)$$

$$= \int_1^0 (1-2x+x^2+x^2)\,dx$$

$$= \int_1^0 (1-2x+2x^2)\,dx$$

$$= \left(x-\cancel{2}\cdot\frac{x^2}{\cancel{2}}+2\cdot\frac{x^3}{3}\right)_1^0$$

$$= (0)-\left(\cancel{1}-\cancel{1}+\frac{2}{3}\right) = -\frac{2}{3}$$

Along CA: Equation of line CA is $y-1 = \frac{0-1}{1-0}(x-0)$ i.e. $y = x+1$ so that $dy = dx$ and $0 \le x \le -1$

$$\therefore \int_{CA} (y^2dx - x^2dy) = \int_{x=0}^{-1} (x+1)^2dx - x^2dx = \int_0^{-1} (\cancel{x^2}+2x+1-\cancel{x^2})\,dx$$

$$= \left(\cancel{2}\cdot\frac{x^2}{\cancel{2}}+x\right)_0^{-1}$$

$$= \cancel{1}-\cancel{1}$$

$$= 0$$

$\therefore$ From ①, $\oint_C (y^2dx - x^2dy) = 0 - \frac{2}{3} + 0 = -\frac{2}{3}$

Practice Questions:

1) If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y-4x)\hat{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the curve $C: y = x^3$ in the xy-plane from $(1,1)$ to $(2,8)$ [Ans: 35]

2) Using the line integral, compute the work done by the force $\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$ when it moves a particle from the point $(0,0,0)$ to the point $(2,1,1)$ along the curve $x = 2t^2$, $y = t$, $z = t^3$ [Ans: $8\frac{8}{35}$]

3) Find the total work done by the force $\vec{F} = 3xy\hat{i} - y\hat{j} + 2zx\hat{k}$ in moving a particle around the circle $x^2+y^2=4$. [Ans: 0]

[Hint: Since the circle $C: x^2+y^2=4$ takes place in xy-plane, $z=0$ so that $dz=0$.

The parametric equations of $C: x^2+y^2=4$ are

$x = 2\cos t$, $y = 2\sin t$ so that $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$ and $0 \le t \le 2\pi$.

$\therefore$ Total work done $= \oint_C \vec{F}\cdot d\vec{r} = \oint_C 3xy\,dx - y\,dy \quad (\because z=0)$

$$= \int_0^{2\pi} (-24\sin^2 t\cos t - 4\sin t\cos t)\,dt$$

$$= \left[-24\left(\frac{\sin^3 t}{3}\right) - 2\left(-\frac{\cos 2t}{2}\right)\right]_0^{2\pi}$$

$$= 0$$

Greens theorem

Green's Theorem in the plane :

Statement: If $M(x,y)$, $N(x,y)$, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ be continuous in a region R of the xy-plane bounded by a closed curve C, then $\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$, where C is traversed in anti-clockwise (positive) direction.

Example 1: Verify Green's theorem for $\oint_C [(xy+y^2)dx + x^2 dy]$, where C is bounded by $y = x$ and $y = x^2$.

Solution: By Green's theorem,

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

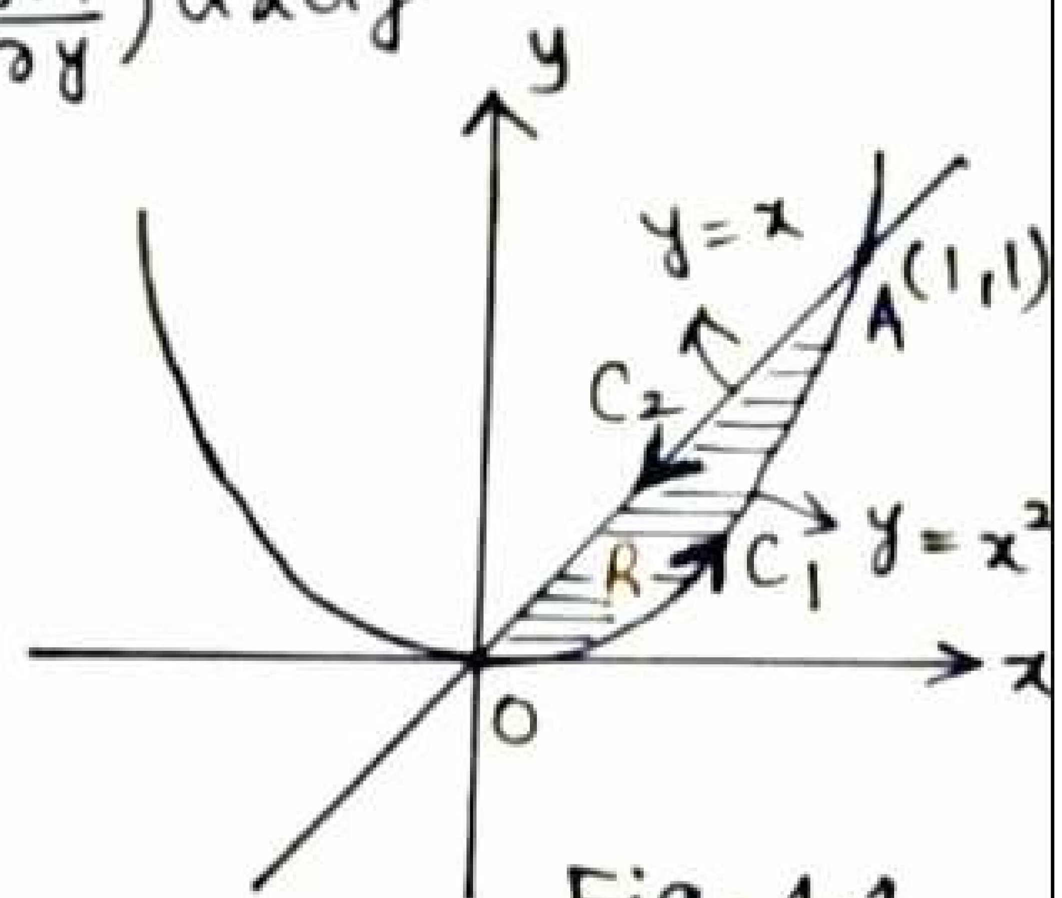


Fig. 1.1

Here $M = xy + y^2$ and $N = x^2$

$\therefore \frac{\partial M}{\partial y} = x + 2y$ and $\frac{\partial N}{\partial x} = 2x$

For the region R: y varies from x^2 to x and x varies from 0 to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[\int_{y=x^2}^{x} (x-2y)\,dy\right] dx$$

$$= \int_{x=0}^{1} \left[x(y) - \cancel{2}\left(\frac{y^2}{\cancel{2}}\right)\right]_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[\{x(x) - x^2\} - \{x(x^2) - (x^2)^2\}\right] dx$$

$$= \int_0^1 (\cancel{x^2} - \cancel{x^2} - x^3 + x^4)\,dx$$

i.e., $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \left(-\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1 = -\frac{1}{4} + \frac{1}{5} = -\frac{1}{20} \quad \text{—(i)}$$

we have $\oint_C (M\,dx + N\,dy) = \int_{C_1} + \int_{C_2}$ —— (ii)

Along C_1: $y = x^2$ so that $dy = 2x\,dx$ and x varies from 0 to 1 (Fig. 1.1)

$$\therefore \int_{C_1} (M\,dx + N\,dy) = \int_{C_1} [(xy + y^2)dx + x^2 dy]$$

$$= \int_{x=0}^{1} [x(x^2) + (x^2)^2]dx + x^2(2x)\,dx$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= \left(3 \cdot \frac{x^4}{4} + \frac{x^5}{5}\right)_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along C_2: $y = x$ so that $dy = dx$ and x varies from 1 to 0 (Fig 1.1)

$$\therefore \int_{C_2} (M\,dx + N\,dy) = \int_{C_2} [(xy + y^2)dx + x^2 dy]$$

$$= \int_{x=1}^{0} [x(x) + x^2]dx + x^2 dx$$

$$= \int_1^0 (x^2 + x^2 + x^2)\,dx$$

$$= 3\int_1^0 x^2\,dx = \cancel{3}\left(\frac{x^3}{\cancel{3}}\right)_1^0 = -1$$

$\therefore$ From (ii), $\oint_C (M\,dx + N\,dy) = \frac{19}{20} - 1 = -\frac{1}{20}$ —— (iii)

From (i) & (iii), LHS = RHS

Hence Green's theorem is verified.

Example 2: Verify Green's theorem for $\oint_C [(3x - 8y^2)dx + (4y - 6xy)dy]$, where C is the boundary of the region enclosed by $x = 0$, $y = 0$ and $x + y = 1$

Solution: By Green's theorem, ③

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

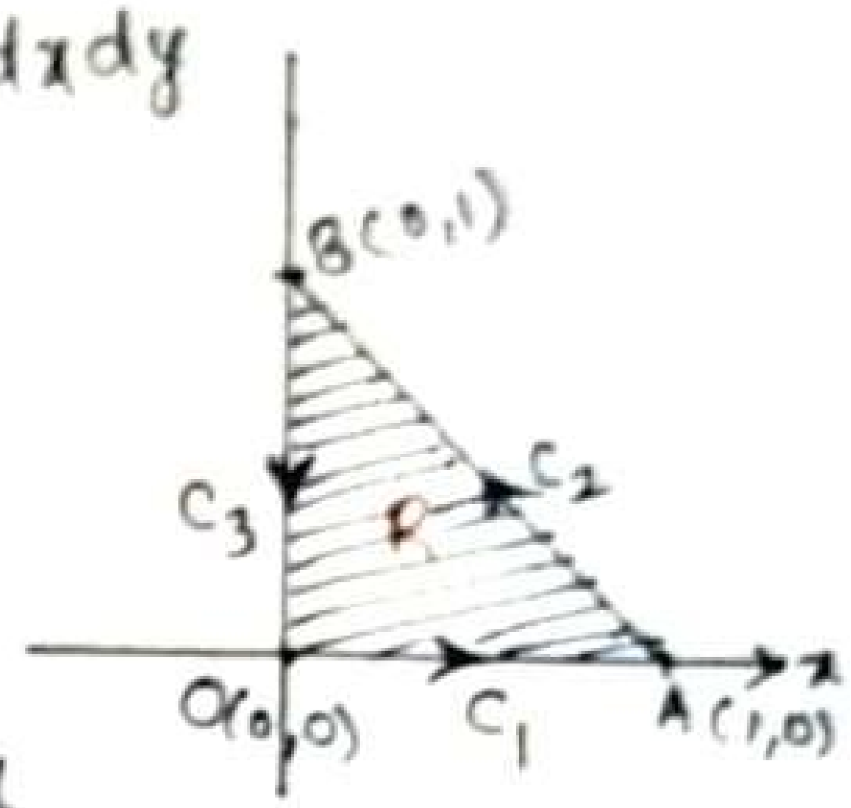


Fig. 1.2

Here $M = 3x - 8y^2$ and $N = 4y - 6xy$

$$\therefore \frac{\partial M}{\partial y} = -16y \quad \text{and} \quad \frac{\partial N}{\partial x} = -6y$$

For the region R: y varies from 0 to $1-x$ and x varies from 0 to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-6y + 16y)\, dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (10y)\, dy\, dx$$

$$= 10 \int_{x=0}^{1} \left[\int_{y=0}^{1-x} y\, dy \right] dx$$

$$= 10 \int_{x=0}^{1} \left[\frac{y^2}{2} \right]_{y=0}^{1-x} dx$$

$$= \cancel{10}^{\,5} \int_0^1 \frac{(1-x)^2}{\cancel{2}}\, dx$$

$$= 5 \int_0^1 (1 - 2x + x^2)\, dx$$

$$= 5 \left(x - \cancel{2} \cdot \frac{x^2}{\cancel{2}} + \frac{x^3}{3} \right)_0^1$$

$$= 5 \left(\cancel{1} - \cancel{1} + \frac{1}{3} \right)$$

i.e., $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{5}{3} \quad \text{———— (i)}$$

We have $$\oint_C (M\,dx + N\,dy) = \int_{C_1} + \int_{C_2} + \int_{C_3} \quad \text{———— (ii)}$$ [Fig. 1.2]

Along C_1: $y = 0$ so that $dy = 0$ and x varies from 0 to 1

$$\therefore \int_{C_1} (M\,dx + N\,dy) = \int_{C_1} (3x - 8y^2)\, dx + (4y - 6xy)\, dy$$

$$= \int_{x=0} (3x - 0)\,dx + 0$$

$$= 3\int_0^1 x\,dx = 3\left(\frac{x^2}{2}\right)_0^1 = \frac{3}{2}$$

Along C_2: $x + y = 1$ i.e., $y = 1 - x$ so that $dy = -dx$ and x varies from 1 to 0.

$$\therefore \int_{C_2} (M\,dx + N\,dy) = \int_{C_2} (3x - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_{x=1}^{0} \left(3x - 8(1-x)^2\right)dx + \left(4(1-x) - 6x(1-x)\right)(-dx)$$

$$= \int_1^0 \left[3x - 8(1-x)^2 - 4(1-x) + 6x(1-x)\right]dx$$

$$= \int_1^0 (3x - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)\,dx$$

$$= \int_1^0 (29x - 14x^2 - 12)\,dx$$

$$= \left(29\cdot\frac{x^2}{2} - 14\cdot\frac{x^3}{3} - 12x\right)_1^0$$

$$= 0 - \left(\frac{29}{2} - \frac{14}{3} - 12\right)$$

$$= \frac{13}{6}$$

Along C_3: $x = 0$ so that $dx = 0$ and y varies from 1 to 0

$$\therefore \int_{C_3} (M\,dx + N\,dy) = \int_{C_3} (3x - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_{y=1}^{0} \left[0 + (4y - 0)\,dy\right]$$

$$= 4\int_1^0 y\,dy$$

$$= 4\left(\frac{y^2}{2}\right)_1^0$$

$$= 2(0 - 1)$$

$$= -2$$

$\therefore$ From (ii), $\oint_C (M\,dx + N\,dy) = \frac{3}{3} + \frac{13}{6} - 2 = \frac{5}{3}$ —— (iii)

From (i) & (iii), LHS = RHS

Hence Green's theorem is verified.

Example 3: Verify Green's theorem for $\oint_C [(x^2 - \cosh y)\,dx + (y + \sin x)\,dy]$, where C is the rectangle with vertices $(0,0), (\pi,0), (\pi,1), (0,1)$.

Solution: By Green's theorem,

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

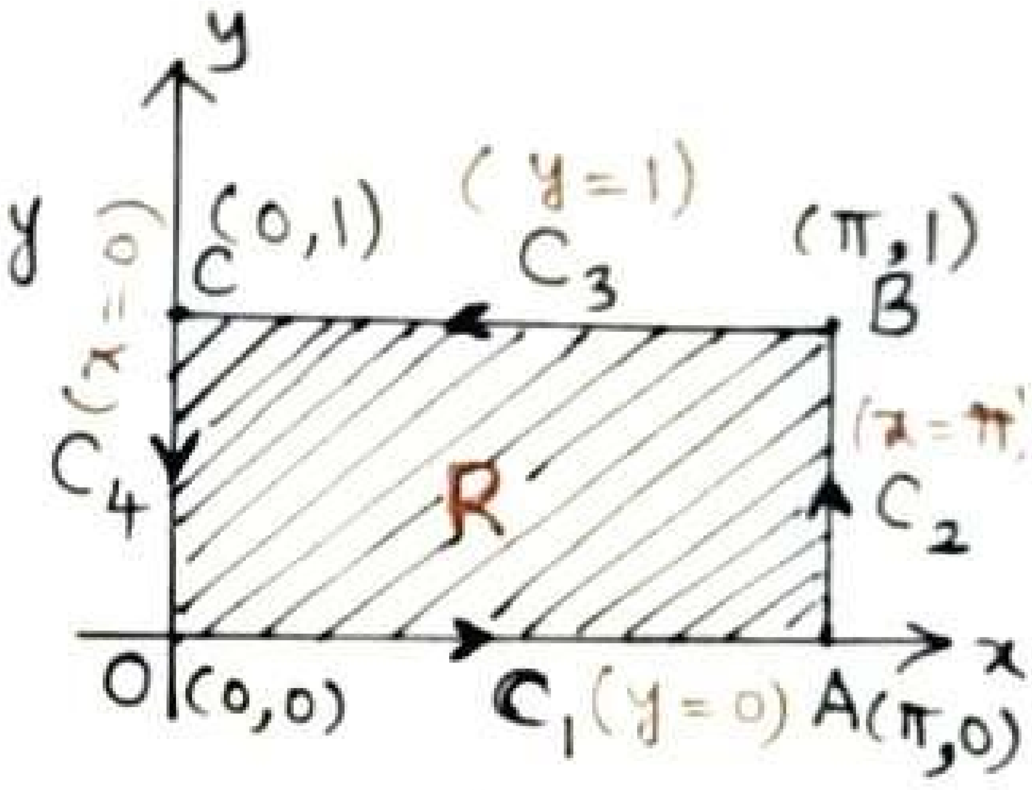


Fig. 1.3

Here $M = (x^2 - \cosh y)$ and $N = (y + \sin x)$

$\therefore \frac{\partial M}{\partial y} = -\sinh y$ and $\frac{\partial N}{\partial x} = \cos x$

For the region R : x varies from 0 to π and y varies from 0 to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (\cos x + \sinh y)\,dx\,dy$$

$$= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\,dy\,dx$$

$$= \int_{x=0}^{\pi} \left[\int_{y=0}^{1} (\cos x + \sinh y)\,dy\right] dx$$

$$= \int_{x=0}^{\pi} \Big[\cos x\,(y) + \cosh y\Big]_{y=0}^{1} dx$$

$$= \int_0^{\pi} \left[(\cos x + \cosh 1) - (0 + \cosh 0)\right] dx$$

$$= \int_0^{\pi} (\cos x + \cosh 1 - 1)\,dx \quad (\because \cosh 0 = 1)$$

$$= \left[\sin x + (\cosh 1 - 1)\,x\right]_0^{\pi}$$

$$= \sin\pi + (\cosh 1 - 1)\pi - 0 \qquad (\because \sin\pi = 0)$$

i.e., $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \pi(\cosh 1 - 1) \text{ ——— (i)}$$

we have $$\oint_C (M\,dx + N\,dy) = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \text{ ——— (ii)} \quad [\because \text{Fig } 1\cdot 3]$$

Along C_1: $y = 0$ so that $dy = 0$ and x varies from 0 to π

$$\therefore \int_{C_1} (M\,dx + N\,dy) = \int_{x=0}^{\pi} \left[(x^2 - \cosh 0)\,dx + (0 + \sin x)(0)\right]$$

$$= \int_0^{\pi} (x^2 - 1)\,dx \qquad \left(\because \cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1\right)$$

$$= \left(\frac{x^3}{3} - x\right)_0^{\pi} = \frac{\pi^3}{3} - \pi$$

Along C_2: $x = \pi$ so that $dx = 0$ and y varies from 0 to 1

$$\therefore \int_{C_2} (M\,dx + N\,dy) = \int_{y=0}^{1} \left[(\pi^2 - \cosh y)(0) + (y + \sin\pi)\,dy\right]$$

$$= \int_0^1 y\,dy \qquad (\because \sin\pi = 0)$$

$$= \left(\frac{y^2}{2}\right)_0^1 = \frac{1}{2}$$

Along C_3: $y = 1$ so that $dy = 0$ and x varies from π to 0

$$\therefore \int_{C_3} (M\,dx + N\,dy) = \int_{x=\pi}^{0} \left[(x^2 - \cosh 1)\,dx + (1 + \sin x)(0)\right]$$

$$= \int_{\pi}^{0} (x^2 - \cosh 1)\,dx$$

$$= \left(\frac{x^3}{3} - x\cosh 1\right)_{\pi}^{0}$$

$$= 0 - \left(\frac{\pi^3}{3} - \pi\cosh 1\right)$$

$$= -\frac{\pi^3}{3} + \pi\cosh 1$$

Along C_4: $x = 0$ so that $dx = 0$ and y varies from 1 to 0.

$$\therefore \int_{C_4} (M\,dx + N\,dy) = \int_{y=1}^{0} \left[(0 - \cosh y)(0) + (y + \sin 0)\,dy \right]$$

$$= \int_{1}^{0} y\,dy = \left(\frac{y^2}{2} \right)_{1}^{0} = -\frac{1}{2}$$

From (ii), $$\oint_C (M\,dx + N\,dy) = \cancel{\frac{\pi^3}{3}} - \pi + \cancel{\frac{1}{2}} - \cancel{\frac{\pi^3}{3}} + \pi\cosh 1 - \cancel{\frac{1}{2}}$$

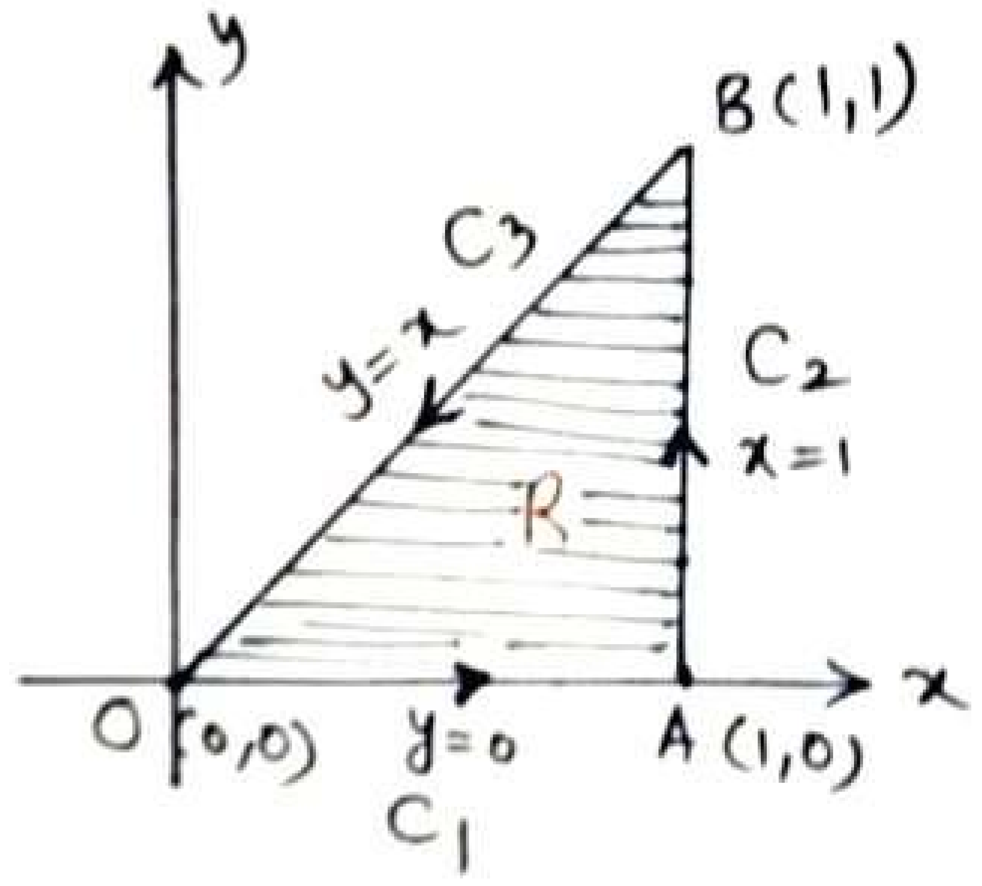
$$= \pi(\cosh 1 - 1) \quad \text{———— (iii)}$$

From (i) & (iii), LHS = RHS

Hence Green's theorem is verified.

HW

Example 4. Verify Green's theorem for $\oint_C (x^2 y\,dx + x^2\,dy)$, where C is the boundary described counter clockwise of triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. [Answer: $\frac{5}{12}$]



Fourier series

Fourier Series

Euler's Formulae: The Fourier series for the function $f(x)$ in the interval $c \le x \le c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

where
$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$
$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$
$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

These values of a_0, a_n, b_n are Known as Euler's formulae.

Some important results:

* $\cos n\pi = (-1)^n,\ n \in \mathbb{Z}$ * $\cos 2n\pi = \cos 2(n\pm1)\pi = 0,\ n \in \mathbb{Z}$
* $\sin n\pi = \sin 2n\pi = \sin 2(n\pm1)\pi = 0,\ n \in \mathbb{Z}$
* $\cos(2n\pm1)\pi = -1,\ n \in \mathbb{Z}$ * $\sin(2n\pm1)\pi = 0,\ n \in \mathbb{Z}$
* $\cos(2n+1)\frac{\pi}{2} = 0,\ n \in \mathbb{Z}$ * $\sin(2n+1)\frac{\pi}{2} = (-1)^n,\ n \in \mathbb{Z}$
* $\cos(n\pm1)\pi = (-1)^{n\pm1},\ n \in \mathbb{Z}$ * $\sin(n\pm1)\pi = 0,\ n \in \mathbb{Z}$

* $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$
* $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$
* **Leibnitz's Rule:** $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

where superscript(′) denotes the differentiation and Subscripts(1,2,3,4) denotes the integration w.r.t. x.

e.g: $\int \underset{u}{x^3}\,\underset{v}{\cos nx}\,dx = x^3\left(\frac{\sin nx}{n}\right) - 3x^2\left(-\frac{\cos nx}{n^2}\right) + 6x\left(-\frac{\sin nx}{n^3}\right) - 6\left(\frac{\cos nx}{n^4}\right)$

Note: To apply Leibnitz's rule, take u as a polynomial function.

Example 1. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

Solution: Given $f(x) = e^{-x}$

Here $c = 0$ and $c + 2l = 2\pi \Rightarrow l = \pi$

The Fourier Series for the function in the interval $c < x < c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right] \quad \text{——— (1)}$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left(-e^{-x}\right)_0^{2\pi} = \frac{1}{\pi}\left(1 - e^{-2\pi}\right)$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx \qquad (\because l = \pi)$$

$$= \frac{1}{\pi}\cdot\left\{\frac{e^{-x}}{(1+n^2)}\left[-\cos nx + n\sin nx\right]\right\}_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}\left(-\cos 2n\pi + n\sin 2n\pi\right) - e^{-0}\left(-\cos 0 + n\sin 0\right)\right]$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}(-1+0) - (-1+0)\right] \quad \left(\begin{array}{l}\because \cos 2n\pi = 1,\ n \in \mathbb{Z} \\ \text{and } \sin 2n\pi = 0,\ n \in \mathbb{Z}\end{array}\right)$$

i.e., $a_n = \dfrac{1 - e^{-2\pi}}{\pi(n^2+1)}$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right)dx$$

$$= \frac{1}{\pi}\cdot\left\{\frac{e^{-x}}{(1+n^2)}\left[-\sin nx - n\cos nx\right]\right\}_0^{2\pi}$$

③

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}\left(-\sin 2n\pi - n\cos 2n\pi\right) - e^{-0}\left(-\sin 0 - n\cos 0\right)\right]$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}\left(-0-n\right) - \left(-0-n\right)\right] \quad \left(\begin{array}{l}\because \sin 2n\pi = 0,\ n\in Z \\ \text{and } \cos 2n\pi = 1,\ n\in Z\end{array}\right)$$

i.e., $b_n = \dfrac{n(1-e^{-2\pi})}{\pi(n^2+1)}$

Substituting a_0, a_n, b_n values in ①, we get

$$e^{-x} = \frac{(1-e^{-2\pi})}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(1-e^{-2\pi})}{\pi(n^2+1)}\cos nx + \frac{n(1-e^{-2\pi})}{\pi(n^2+1)}\sin nx\right]$$

or $$e^{-x} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2+1} + \frac{n\sin nx}{n^2+1}\right)\right]$$

Example 2. Find a Fourier series to represent $x-x^2$ from $x=-\pi$ to $x=\pi$. Hence deduce that $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \frac{\pi^2}{12}$.

Solution. Let $f(x) = x-x^2$

Here $c=-\pi$ and $c+2l=\pi \Rightarrow l=\pi$

The Fourier series for the function $f(x)$ in the interval $c \le x \le c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right) \quad \text{—— ①}$$

where $$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx$$

$$= \frac{1}{\pi}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_{-\pi}^{\pi}$$

i.e., $$a_0 = \frac{1}{\pi}\left[\left(\cancel{\frac{\pi^2}{2}}-\frac{\pi^3}{3}\right)-\left(\cancel{\frac{\pi^2}{2}}+\frac{\pi^3}{3}\right)\right] = -\frac{2}{3}\pi^2$$

(4)

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\Big[\int_{-\pi}^{\pi}\underbrace{x\cos nx}_{\text{(odd } f^n)}\,dx - \int_{-\pi}^{\pi}\underbrace{x^2\cos nx}_{\text{(even } f^n)}\,dx\Big]$$

$$= \frac{1}{\pi}\Big[0 - 2\cdot\int_{0}^{\pi}\underbrace{x^2}_{u}\underbrace{\cos nx}_{v}\,dx\Big] \quad \left(\because \int_{-a}^{a}f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(-x)=f(x) \\ 0 & \text{if } f(-x)=-f(x)\end{cases}\right)$$

$$= -\frac{2}{\pi}\Big[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\Big]_0^{\pi}$$

[$\because$ By applying Leibnitz's Rule]

$$= -\frac{2}{\pi}\Big[\pi^2\left(\frac{\sin n\pi}{n}\right) + 2\pi\left(\frac{\cos n\pi}{n^2}\right) - 2\left(\frac{\sin n\pi}{n^3}\right) - 0\Big]$$

$$= -\frac{2}{\pi}\Big[\frac{\pi^2}{n}(0) + \frac{2\pi}{n^2}(-1)^n - \frac{2}{n^3}(0)\Big] \quad \left(\because \sin n\pi = 0 \ \& \ \cos n\pi = (-1)^n \text{ for } n\in\mathbb{Z}\right)$$

$$= -\frac{2}{\cancel{\pi}}\Big[\frac{2\cancel{\pi}}{n^2}(-1)^n\Big]$$

i.e., $a_n = \frac{4}{n^2}(-1)^{n+1}$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\Big[\int_{-\pi}^{\pi}\underbrace{x\sin nx}_{\text{(even } f^n)}\,dx - \int_{-\pi}^{\pi}\underbrace{x^2\sin nx}_{\text{(odd } f^n)}\,dx\Big]$$

$$= \frac{1}{\pi}\Big[2\int_0^{\pi}x\sin nx\,dx - 0\Big]$$

$$= \frac{2}{\pi}\Big[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\Big[-\frac{\pi}{n}\cos n\pi + \frac{\sin n\pi}{n^2} + 0\Big]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n + 0\right]$$

i.e., $b_n = \frac{2}{\pi}\left[\frac{\pi}{n}(-1)^{n+1}\right] = \frac{2}{n}(-1)^{n+1}$

Substituting a_0, a_n, b_n values in ①, we get

$$x - x^2 = \frac{1}{2}\left(-\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^{n+1}\cos nx + \frac{2}{n}(-1)^{n+1}\sin nx\right]$$

i.e., $x - x^2 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$ —— ②

put $x = 0$ in ②, we get

$$0 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos 0 + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin 0$$

$$\Rightarrow 0 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} + 2(0)$$

$$\Rightarrow 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

Example 3. If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the range 0 to 2π, show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$.

Solution: Here $(c, c+2l) = (0, 2\pi) \Rightarrow c = 0$ and $c + 2l = 2\pi$

$\Rightarrow l = \pi$

The Fourier series for $f(x)$ in the interval $(c, c+2l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad \text{—— ①}$$

(6)

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2 dx$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi^2-2\pi x+x^2)\,dx$$

$$= \frac{1}{4\pi}\left(\pi^2 x - \cancel{2}\pi\cdot\frac{x^2}{\cancel{2}}+\frac{x^3}{3}\right)_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\pi^2(2\pi) - \pi(2\pi)^2 + \frac{(2\pi)^3}{3} - 0\right]$$

$$= \frac{1}{4\pi}\left(2\pi^3 - 4\pi^3 + \frac{8\pi^3}{3}\right)$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\cdot\left(\frac{\sin nx}{n}\right) - 2(\pi-x)(-1)\cdot\left(-\frac{\cos nx}{n^2}\right) + 2(1)\cdot\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi-2\pi)^2\cdot\frac{\sin 2n\pi}{n} - 2(\pi-2\pi)\cdot\frac{\cos 2n\pi}{n^2} - \frac{2}{n^3}\sin 2n\pi\right]$$

$$- \frac{1}{4\pi}\left[\pi^2\cdot\left(\frac{\sin 0}{n}\right) - 2(\pi-0)\frac{\cos 0}{n^2} - \frac{2}{n^3}\sin 0\right]$$

$$= \frac{1}{4\pi}\left[0 - 2(-\pi)\cdot\frac{1}{n^2} - 0\right] - \frac{1}{4\pi}\left[0 - \frac{2\pi}{n^2} - 0\right] \quad \left[\begin{array}{l}\because \cos 2n\pi = 1 \\ \text{\&}\ \sin 2n\pi = 0\end{array}\right]$$

i.e., $a_n = \frac{1}{n^2}$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2\sin\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi-x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(1)\cdot\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$ 7

$$= \frac{1}{4\pi}\left[(\pi-2\pi)^2\left(-\frac{\cos 2n\pi}{n}\right) - 2(\pi-2\pi)\frac{\sin 2n\pi}{n^2} + \frac{2}{n^3}\cos 2n\pi\right]$$

$$-\frac{1}{4\pi}\left[(\pi-0)^2\left(-\frac{\cos 0}{n}\right) - 2(\pi-0)\left(\frac{\sin 0}{n^2}\right) + \frac{2}{n^3}\cos 0\right]$$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n}(1) - 0 + \frac{2}{n^3}(1)\right] - \frac{1}{4\pi}\left[-\frac{\pi^2}{n} - 0 + \frac{2}{n^3}\right]$$

$$\text{i.e } b_n = \cancel{-\frac{\pi}{4n}} + \cancel{\frac{1}{2n^3\pi}} + \cancel{\frac{\pi}{4n}} - \cancel{\frac{1}{2n^3\pi}} = 0$$

Substituting a_0, a_n, b_n values in ①, we get

$$f(x) = \frac{1}{2}\left(\frac{\pi^2}{6}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos nx + (0)\sin nx\right)$$

$$\text{i.e., } f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

Example 4. Expand $f(x) = \sqrt{1-\cos x}$, $0 < x < 2\pi$ in a Fourier Series. Hence evaluate $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$

Solution: Given $f(x) = \sqrt{1-\cos x} = \sqrt{2\sin^2\frac{x}{2}} = \sqrt{2}\cdot\sin x/2$

Here $(c, c+2l) = (0, 2\pi) \Rightarrow c = 0$ & $c+2l = 2\pi$

$\Rightarrow l = \pi$

The Fourier series for $f(x)$ in $(c, c+2l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad \text{——— ①}$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\cdot\sin x/2\,dx$

$$= \frac{\sqrt{2}}{\pi}\left(-\frac{\cos x/2}{1/2}\right)_0^{2\pi} = -\frac{2\sqrt{2}}{\pi}\left(\cos\frac{2\pi}{2} - \cos 0\right)$$

$$= -\frac{2\sqrt{2}}{\pi}(-1-1)$$

i.e., $a_0 = \dfrac{4\sqrt{2}}{\pi}$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\cdot \sin x/2 \cdot \cos\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\cos nx \sin\frac{x}{2}\,dx$$

$$= \frac{1}{\pi\sqrt{2}}\int_0^{2\pi}\left[\sin(n+\tfrac{1}{2})x - \sin(n-\tfrac{1}{2})x\right]dx$$

$$= \frac{1}{\pi\sqrt{2}}\left[\int_0^{2\pi}\sin(2n+1)\frac{x}{2}\,dx - \int_0^{2\pi}\sin(2n-1)\frac{x}{2}\,dx\right]$$

$$= \frac{1}{\pi\sqrt{2}}\left[-\frac{\cos(2n+1)\frac{x}{2}}{\left(\frac{2n+1}{2}\right)} + \frac{\cos(2n-1)\frac{x}{2}}{\left(\frac{2n-1}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{1}{\pi\sqrt{2}}\left[-\frac{2}{2n+1}\cos(2n+1)\pi + \frac{2}{2n-1}\cos(2n-1)\pi + \frac{2}{2n+1}\cos 0 - \frac{2}{2n-1}\cos 0\right]$$

$$= \frac{1}{\pi\sqrt{2}}\left[-\frac{2}{2n+1}(-1) + \frac{2}{2n-1}(-1) + \frac{2}{2n+1} - \frac{2}{2n-1}\right]$$

$$= \frac{4}{\pi\sqrt{2}}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$= \frac{4}{\pi\sqrt{2}}\cdot\frac{(\cancel{2n}-1-\cancel{2n}-1)}{(2n+1)(2n-1)}$$

i.e., $a_n = \dfrac{-8}{\pi\sqrt{2}(4n^2-1)}$

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$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\cdot\sin\frac{x}{2}\,\sin\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right)dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin nx\sin\frac{x}{2}\,dx$$

$$= \frac{1}{\pi\sqrt{2}}\int_0^{2\pi}\left[\cos(n-\tfrac{1}{2})x - \cos(n+\tfrac{1}{2})x\right]dx$$

$$= \frac{1}{\pi\sqrt{2}}\int_0^{2\pi}\left[\cos(2n-1)\frac{x}{2} - \cos(2n+1)\frac{x}{2}\right]dx$$

$$= \frac{1}{\pi\sqrt{2}}\left[\frac{\sin(2n-1)\frac{x}{2}}{\left(\frac{2n-1}{2}\right)} - \frac{\sin(2n+1)\frac{x}{2}}{\left(\frac{2n+1}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{1}{\pi\sqrt{2}}\left[\frac{2}{2n-1}\cdot\sin(2n-1)\pi - \frac{2}{2n+1}\sin(2n+1)\pi\right] - \frac{1}{\pi\sqrt{2}}(0)$$

$$= \frac{1}{\pi\sqrt{2}}\left[\frac{2}{2n-1}(0) - \frac{2}{2n+1}(0)\right] \quad (\because \sin(2n\pm1)\pi = 0,\ n\in\mathbb{Z})$$

i.e., $b_n = 0$

Substituting a_0, a_n, b_n values in ①, we get

$$\sqrt{1-\cos x} = \frac{1}{2}\left(\frac{4\sqrt{2}}{\pi}\right) + \sum_{n=1}^{\infty}\left[\frac{-8}{\pi\sqrt{2}(4n^2-1)}\cos nx + (0)\sin nx\right]$$

i.e., $$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{4n^2-1} \quad \text{—— ②}$$

put $x=0$ in ②, we get

$$\sqrt{1-\cos 0} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos 0}{4n^2-1}$$

i.e., $$\sqrt{1-1} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\Rightarrow \frac{\overset{2}{\cancel{4}}\cancel{\sqrt{2}}}{\cancel{\pi}}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{\cancel{2}\cancel{\sqrt{2}}}{\cancel{\pi}}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots=\frac{1}{2}$$

Example 5. Expand $f(x)=x\sin x$ as a Fourier series in $0<x<2\pi$

Solution: Here $(c, c+2l)=(0, 2\pi)$

$$\Rightarrow c=0 \quad \text{and} \quad c+2l=2\pi$$

$$\Rightarrow l=\pi$$

The Fourier series for $f(x)$ in $(c, c+2l)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right)+b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right) \quad \text{—— (1)}$$

where $a_0=\frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$$=\frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$=\frac{1}{\pi}\Big[x(-\cos x)-(1)(-\sin x)\Big]_0^{2\pi} \quad \text{[By Leibnitz's Rule]}$$

$$=\frac{1}{\pi}\Big[-2\pi\cos 2\pi+\sin 2\pi+0\Big]$$

i.e., $a_0=\frac{1}{\pi}\left[-2\pi(1)+0\right]=-2$

$$a_n=\frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x\sin x\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x\left(2\cos nx\sin x\right)dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} \underset{u}{x}\Big[\underbrace{\sin(n+1)x-\sin(n-1)x}_{v}\Big]dx$$

$$=\frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right\}-(1)\left\{-\frac{\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1}+\frac{\cos 2(n-1)\pi}{n-1}\right\}-\left\{-\frac{\sin 2(n+1)\pi}{(n+1)^2}+\frac{\sin 2(n-1)\pi}{(n-1)^2}\right\}\right]$$

$$-\frac{1}{2\pi}\left[0-\left\{-\frac{\sin 0}{(n+1)^2}+\frac{\sin 0}{(n-1)^2}\right\}\right]$$

$$= \frac{1}{2\cancel{\pi}}\left[2\cancel{\pi}\left\{-\frac{1}{n+1}+\frac{1}{n-1}\right\}-\left\{0\right\}\right]-\frac{1}{2\pi}(0) \quad (n\neq 1)$$

$$= \frac{1}{n-1}-\frac{1}{n+1} \quad (n\neq 1)$$

$$= \frac{n+1-n+1}{(n-1)(n+1)} \quad (n\neq 1)$$

i.e., $a_n = \dfrac{2}{n^2-1} \quad (n\neq 1)$

When $n=1$, $a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} x\sin x\cos x\,dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-(1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\cancel{2}\pi\cdot\frac{\cos 4\pi}{\cancel{2}}+\frac{\sin 4\pi}{4}+0\right]$$

i.e., $a_1 = \dfrac{1}{2\pi}\left[-\pi(1)+\dfrac{0}{4}\right] = -\dfrac{1}{2}$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx\sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x-\cos(n+1)x\right]dx$$

(12)

$$= \frac{1}{2\pi}\left[x\left\{ \frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1} \right\} - (1)\left\{ -\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2} \right\} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{ \frac{\sin 2(n-1)\pi}{n-1} - \frac{\sin 2(n+1)\pi}{n+1} \right\} - \left\{ -\frac{\cos 2(n-1)\pi}{(n-1)^2} + \frac{\cos 2(n+1)\pi}{(n+1)^2} \right\} \right]$$

$$- \frac{1}{2\pi}\left[0 - \left\{ -\frac{\cos 0}{(n-1)^2} + \frac{\cos 0}{(n+1)^2} \right\} \right]$$

$$= \frac{1}{2\pi}\left[2\pi\{0\} - \left\{ -\frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} \right\} \right] - \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} \right] \quad (n \neq 1)$$

$$= \frac{1}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} - \frac{1}{(n+1)^2} \right] - \frac{1}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} - \frac{1}{(n+1)^2} \right] \quad (n \neq 1)$$

i.e., $b_n = 0 \quad (n \neq 1)$

when $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin^2 x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\, dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos 2x\, dx$$

$$= \frac{1}{2\pi}\left(\frac{x^2}{2}\right)_0^{2\pi} - \frac{1}{2\pi}\left[x\left(\frac{\sin 2x}{2}\right) - (1)\left(-\frac{\cos 2x}{4}\right) \right]_0^{2\pi}$$

$$= \frac{1}{4\pi}(2\pi)^2 - \frac{1}{2\pi}\left[2\pi\left(\frac{\sin 4\pi}{2}\right) + \frac{\cos 4\pi}{4} - 0 - \frac{\cos 0}{4} \right]$$

$$= \frac{\cancel{4}\pi^{\cancel{2}}}{\cancel{4\pi}} - \frac{1}{2\pi}\left[2\pi(0) + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{4}} \right]$$

i.e., $b_1 = \pi$

Substituting a_0, a_n, b_n values in (1), we get

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} (a_n \cos nx + b_n \sin nx)$$

i.e., $x \sin x = \frac{1}{2}(-2) + (-\frac{1}{2}) \cos x + \pi \sin x + \sum_{n=2}^{\infty} \left(\frac{2}{n^2-1} \cos nx + (0) \sin nx \right)$

or $x \sin x = -1 - \frac{1}{2} \cos x + \pi \sin x + 2 \sum_{n=2}^{\infty} \frac{\cos nx}{n^2-1}$

Example 6. If $f(x) = 2x - x^2$ in $0 \le x \le 2$, show that

$$f(x) = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n}$$

Solution:

Here $[c, c+2l] = [0, 2]$

$\Rightarrow c = 0$ and $c + 2l = 2$

$\Rightarrow l = 1$

The Fourier series for $f(x)$ in $c \le x \le c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\pi x + b_n \sin n\pi x \right) \quad \text{—— ①} \left(\because l = 1\right)$$

where $a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\,dx$

$$= \frac{1}{1} \int_0^2 (2x - x^2)\,dx$$

$$= \left(\cancel{2} \cdot \frac{x^2}{\cancel{2}} - \frac{x^3}{3} \right)_0^2$$

i.e., $a_0 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{1} \int_0^2 (x - x^2) \cos\left(\frac{n\pi x}{1}\right) dx$$

$$= \int_0^2 (\underbrace{x-x^2}_{u}) \cos \underset{v}{n\pi x}\, dx \quad \text{[By applying Leibnitz's Rule]}$$ (14)

$$= \left[(x-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (0-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right) \right]_0^2$$

$$= \left[(2-4)\left(\frac{\sin 2n\pi}{n\pi}\right) + (1-4)\left(\frac{\cos 2n\pi}{n^2\pi^2}\right) + \frac{2}{n^3\pi^3}(\sin 2n\pi) \right]$$

$$- \left[0 + (1-0)\frac{\cos 0}{n^2\pi^2} + 2(0) \right]$$

$$= (-2)(0) - \frac{3}{n^2\pi^2}(1) + \frac{2}{n^3\pi^3}(0) - \frac{1}{n^2\pi^2} \quad \left(\begin{array}{l} \because \sin 2n\pi = 0,\ n \in \mathbb{Z} \\ \text{and } \cos 2n\pi = 1,\ n \in \mathbb{Z} \end{array} \right)$$

i.e., $a_n = -\dfrac{4}{n^2\pi^2}$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{1}\int_0^2 (x-x^2) \sin\left(\frac{n\pi x}{1}\right) dx$$

$$= \int_0^2 (\underbrace{x-x^2}_{u}) \sin \underset{v}{n\pi x}\, dx \quad \text{[By applying Leibnitz's Rule]}$$

$$= \left[(x-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right) \right]_0^2$$

$$= \left[(2-4)\left(-\frac{\cos 2n\pi}{n\pi}\right) + (1-4)\left(\frac{\sin 2n\pi}{n^2\pi^2}\right) - \frac{2}{n^3\pi^3}\cos 2n\pi \right]$$

$$- \left[0 + 0 - \frac{2}{n^3\pi^3}\cos 0 \right]$$

$$= (-2)\left(-\frac{1}{n\pi}\right) - 3(0) - \cancel{\frac{2}{n^3\pi^3}}(1) + \cancel{\frac{2}{n^3\pi^3}}$$

i.e., $b_n = \dfrac{2}{n\pi}$

Substituting a_0, a_n, b_n values in (1), we get

$$f(x) = \frac{1}{2}\left(\frac{4}{3}\right) + \sum_{n=1}^{\infty}\left(-\frac{4}{n^2\pi^2}\cos n\pi x + \frac{2}{n\pi}\sin n\pi x\right)$$

i.e., $$f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}$$

Example 7. Find the Fourier series expansion of $f(x) = 2x - x^2$ in $(0, 3)$ and hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

Solution:

Here $(c, c+2l) = (0, 3)$

$\Rightarrow c = 0$ and $c + 2l = 3$

$\Rightarrow l = \frac{3}{2}$

The Fourier series for $f(x)$ in $(c, c+2l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi}{3}x\right) + b_n \sin\left(\frac{2n\pi}{3}x\right)\right] \quad (1) \quad \left(\because l = \frac{3}{2}\right)$$

Where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$

i.e., $$a_0 = \frac{1}{\left(\frac{3}{2}\right)}\int_0^3 (2x - x^2)\,dx = \frac{2}{3}\left(\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^3}{3}\right)_0^3 = 0$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\left(\frac{3}{2}\right)}\int_0^3 (\underbrace{2x - x^2}_{u})\underbrace{\cos\left(\frac{2n\pi}{3}x\right)}_{v}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\cdot\left(\frac{\sin\left(\frac{2n\pi}{3}x\right)}{\frac{2n\pi}{3}}\right) - (2 - 2x)\cdot\left(\frac{-\cos\left(\frac{2n\pi}{3}x\right)}{\left(\frac{2n\pi}{3}\right)^2}\right) + (0 - 2)\left(-\frac{\sin\left(\frac{2n\pi}{3}x\right)}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[(6-9)\left(\frac{3}{2n\pi}\cdot \underset{0}{\sin 2n\pi}\right) + (2-6)\cdot\left(\frac{9}{4n^2\pi^2}\cdot \underset{1}{\cos 2n\pi}\right) + 2\cdot\left(\frac{27}{8n^3\pi^3}\underset{0}{\sin 2n\pi}\right)\right]$$

$$- \frac{2}{3}\left[0 + (2-0)\left(\frac{9}{4n^2\pi^2}\cdot\cos 0\right) + 0\right]$$

$$= \frac{2}{3}\left[0 - 4\left(\frac{9}{4n^2\pi^2}\right) + 0\right] - \frac{2}{3}\left[2\left(\frac{9}{4n^2\pi^2}\right)\right]$$

$$= -\frac{6}{n^2\pi^2} - \frac{3}{n^2\pi^2}$$

i.e., $a_n = -\dfrac{9}{n^2\pi^2}$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\left(\frac{3}{2}\right)}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi}{3}x\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(-\frac{\cos\left(\frac{2n\pi}{3}x\right)}{\frac{2n\pi}{3}}\right) - (2-2x)\left(-\frac{\sin\left(\frac{2n\pi}{3}x\right)}{\left(\frac{2n\pi}{3}\right)^2}\right) + (0-2)\left(\frac{\cos\left(\frac{2n\pi}{3}x\right)}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[(6-9)\left(-\frac{3}{2n\pi}\cdot\underset{1}{\cos 2n\pi}\right) + (2-6)\left(\frac{9}{4n^2\pi^2}\cdot\underset{0}{\sin 2n\pi}\right) - 2\cdot\left(\frac{27}{8n^3\pi^3}\cdot\underset{1}{\cos 2n\pi}\right)\right]$$

$$- \frac{2}{3}\left[0 + (2-0)\cdot\left(\frac{9}{4n^2\pi^2}\cdot\underset{0}{\sin 0}\right) - 2\cdot\left(\frac{27}{8n^3\pi^3}\underset{1}{\cos 0}\right)\right]$$

$$= \frac{2}{3}\left[(-3)\left(-\frac{3}{2n\pi}\right) + 0 - 2\cdot\left(\frac{27}{8n^3\pi^3}\right)\right] - \frac{2}{3}\left[0 - 2\cdot\left(\frac{27}{8n^3\pi^3}\right)\right]$$

$$= \frac{\cancel{2}}{\cancel{3}}\left[\frac{\cancel{9}^{\,3}}{\cancel{2}n\pi} - \cancel{\frac{54}{8n^3\pi^3}} + \cancel{\frac{54}{8n^3\pi^3}}\right]$$

i.e., $b_n = \dfrac{3}{n\pi}$

Substituting a_0, a_n, b_n values in ①, we get

$$2x - x^2 = \frac{1}{2}(0) + \sum_{n=1}^{\infty}\left[-\frac{9}{n^2\pi^2}\cos\left(\frac{2n\pi}{3}x\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi}{3}x\right)\right]$$

or $2x-x^2 = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n\pi}{3}x\right)}{n^2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{2n\pi}{3}x\right)}{n}$ —— ②

put $x=\frac{3}{2}$ in ②, we get

$$\cancel{2}\left(\frac{3}{\cancel{2}}\right)-\left(\frac{3}{2}\right)^2 = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{\cancel{2}n\pi}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{\cancel{2}n\pi}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}\right)$$

i.e., $3-\frac{9}{4} = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\underbrace{\sin n\pi}_{0}$

$$\Rightarrow \frac{3}{4} = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(0)$$

$$\Rightarrow \frac{\overset{3}{\cancel{9}}}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\cancel{3}}{4}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \frac{\pi^2}{12}$$

HW

Example 8. Obtain the Fourier series for $f(x)=\frac{\pi-x}{2}$ in $0\le x\le 2$.

Solution: Here $c=0$ and $c+2l=2 \Rightarrow l=1$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos n\pi x + b_n\sin n\pi x\right)$ —— ①

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{1}\int_0^2\left(\frac{\pi-x}{2}\right)dx = \pi-1$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{1}\int_0^2\left(\frac{\pi-x}{2}\right)\cos n\pi x = 0$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{1}\int_0^2\left(\frac{\pi-x}{2}\right)\sin n\pi x = \frac{1}{n\pi}$$

From ①, we have

$$\frac{(\pi-x)}{2} = \frac{\pi-1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}$$

Fourier Series for functions having points of discontinuity:

Let $f(x)$ be a function defined in $(c, c+2l)$ by

$$f(x) = \begin{cases} \phi(x), & c < x < x_0 \\ \psi(x), & x_0 < x < c+2l \end{cases}$$ i.e., x_0 is the point of discontinuity

then $$a_0 = \frac{1}{l}\int_c^{c+2l} f(x) = \frac{1}{l}\left[\int_c^{x_0} f(x)\,dx + \int_{x_0}^{c+2l} f(x)\,dx\right]$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\left[\int_c^{x_0} f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_{x_0}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\left[\int_c^{x_0} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{x_0}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

Note: $f(x_0) = \frac{1}{2}\left[f(x_0^-) + f(x_0^+)\right] = \frac{1}{2}\left[\phi(x_0) + \psi(x_0)\right]$

Example 1. Find the Fourier Series to represent the function $f(x)$ given by $f(x) = x$ for $0 \le x \le \pi$, and $= 2\pi - x$ for $\pi \le x \le 2\pi$.

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$

Solution. Given $$f(x) = \begin{cases} x, & \underset{c}{\underset{\downarrow}{0}} \le x \le \pi \\ 2\pi - x, & \pi \le x \le \underset{c+2l}{\underset{\downarrow}{2\pi}}. \end{cases}$$

Here $c = 0$ and $c + 2l = 2\pi \Rightarrow l = \pi$

The Fourier series for $f(x)$ in $c \le x \le c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad \text{--- (1)}$$

where $$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - 0\right) + \left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right]$$

i.e., $a_0 = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{4\pi^2}{2} - \frac{3\pi^2}{2}\right] = \pi$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \quad (\because l = \pi)$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \underset{u}{x}\, \underset{v}{\cos nx}\, dx + \int_{\pi}^{2\pi} \underbrace{(2\pi - x)}_{u}\underset{v}{\cos nx}\, dx\right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (0-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{n}\underset{0}{\sin n\pi} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right] + \frac{1}{\pi}\left[\left(0 - \frac{\overset{1}{\cos 2n\pi}}{n^2}\right)\right.$$

$$\left. - \left(\pi\frac{\overset{0}{\sin n\pi}}{n} - \frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \frac{1}{n^2}\right] + \frac{1}{\pi}\left[\left(0 - \frac{1}{n^2}\right) - \left(0 - \frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

i.e., $a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx \quad (\because l = \pi)$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\, dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (0-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi}{n}\cos n\pi + \frac{\overset{0}{\sin n\pi}}{n^2}\right) - (0+0)\right] + \frac{1}{\pi}\left[\left(0 - \frac{\overset{0}{\sin 2n\pi}}{n^2}\right) - \left(-\frac{\pi}{n}\cos n\pi - \frac{\overset{0}{\sin n\pi}}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(-1)^n + 0\right] + \frac{1}{\pi}\left[\frac{\pi}{n}(-1)^n\right]$$

i.e., $b_n = -\cancel{\frac{1}{n}(-1)^n} + \cancel{\frac{1}{n}(-1)^n} = 0$

Substituting a_0, a_n, b_n values in (1), we get

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}[(-1)^n - 1]\cos nx + (0)\sin nx\right]$$

i.e., $f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos nx$ ——— (2)

Put $x = \pi$ in (2), we get

$$f(\pi) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos n\pi$$

i.e., $\pi = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}(-1)^n$ $\left[\begin{array}{l} \because \text{By the definition,} \\ f(x) = x,\ 0 \le x \le \pi \\ \therefore f(\pi) = \pi \end{array}\right]$

$$\Rightarrow \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^{2n} - (-1)^n]}{n^2} = \pi - \frac{\pi}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{[1 - (-1)^n]}{n^2} = \frac{\pi^2}{4} \quad \left(\because (-1)^{2n} = 1\ \forall n \in \mathbb{Z}^+\right)$$

$$\Rightarrow \frac{(1+1)}{1^2} + \frac{(1-1)}{2^2} + \frac{(1+1)}{3^2} + \frac{(1-1)}{4^2} + \frac{(1+1)}{5^2} + \cdots = \frac{\pi^2}{4}$$

$$\Rightarrow 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty\right) = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$$

Example 2. Find the Fourier series expansion for $f(x)$,

if $f(x) = \begin{cases} -\pi, & \underset{c}{-\pi} < x < 0 \\ x, & 0 < x < \underset{c+2l}{\pi} \end{cases}$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Solution: Here $c = -\pi$ and $c + 2l = \pi \Rightarrow l = \pi$

The Fourier Series for $f(x)$ in $c < x < c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \text{——} \; (1)$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{-\pi(x)\right\}_{-\pi}^{0} + \left\{\frac{x^2}{2}\right\}_0^{\pi}\right]$$

i.e., $a_0 = \frac{1}{\pi}\left[0 + (-\pi^2) + \frac{\pi^2}{2} - 0\right] = -\frac{\pi}{2}$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (\because l = \pi)$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^{\pi} \underset{u}{x}\,\underset{v}{\cos nx}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi\cdot\sin nx}{n}\right)_{-\pi}^{0}\right] + \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi}{n}\underset{0}{\underset{\downarrow}{\sin(-n\pi)}}\right] + \frac{1}{\pi}\left[\left(\frac{\pi}{n}\underset{0}{\underset{\downarrow}{\sin n\pi}} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{1}{\pi}(0) + \frac{1}{\pi}\left[\frac{\pi}{n}(0) + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

i.e., $a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \quad (\because l = \pi)$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\, dx + \int_0^{\pi} x\sin nx\, dx\right]$$

$$= \frac{1}{\cancel{\pi}}\left[-\cancel{\pi}\left(-\frac{\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= -\left[-\frac{\cos 0}{n} + \frac{\cos n\pi}{n}\right] + \frac{1}{\pi}\left[\left(-\frac{\pi}{n}\cos n\pi + \frac{\overset{0}{\overset{\uparrow}{\sin n\pi}}}{n^2}\right) - 0\right]$$

$$= \frac{1}{n} - \frac{(-1)^n}{n} + \frac{1}{\cancel{\pi}}\left[-\frac{\cancel{\pi}}{n}(-1)^n + 0\right] \quad (\because \cos n\pi = (-1)^n)$$

$$= \frac{1}{n} - \frac{(-1)^n}{n} - \frac{1}{n}(-1)^n$$

i.e., $b_n = \frac{1}{n}\left[1 - 2(-1)^n\right]$

Substituting a_0, a_n, b_n values in ①, we get

$$f(x) = \frac{1}{2}\left(-\frac{\pi}{2}\right) + \sum_{n=1}^{\infty}\left[\frac{[(-1)^n - 1]}{\pi n^2}\cos nx + \frac{[1 - 2(-1)^n]}{n}\sin nx\right]$$

i.e., $f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{[1 - 2(-1)^n]}{n}\sin nx$ —— ②

put $x = 0$ in ②, we get

$$f(0) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos 0 + \sum_{n=1}^{\infty}\frac{[1 - 2(-1)^n]}{n}\sin 0$$

i.e., $\frac{1}{2}\left[f(0^-)+f(0^+)\right] = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]}{n^2} + \sum_{n=1}^{\infty}\frac{[1-2(-1)^n]}{n}\underset{0}{\underset{\downarrow}{(0)}}$

($\because$ $x=0$ is pt. of discontinuity)

$$\Rightarrow \frac{1}{2}\left[-\pi+0\right] = -\frac{\pi}{4}+\frac{1}{\pi}\left[\frac{(-1-1)}{1^2}+\frac{(1-1)}{2^2}+\frac{(-1-1)}{3^2}+\frac{(1-1)}{4^2}+\cdots\right]$$

$$\Rightarrow -\frac{\pi}{2}+\frac{\pi}{4} = \frac{1}{\pi}\left[\frac{(-2)}{1^2}+\frac{(-2)}{3^2}+\frac{(-2)}{5^2}+\cdots\right]$$

$$\Rightarrow \cancel{+}\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right) = \cancel{+}\frac{\pi}{4}$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\infty = \frac{\pi^2}{8}$$

Example 3. If $f(x) = \begin{cases} 0, & \underset{c}{\underset{\downarrow}{-\pi}} \le x \le 0 \\ \sin x, & 0 \le x \le \underset{c+2l}{\underset{\downarrow}{\pi}}, \end{cases}$ prove that

$f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$. Hence show that $\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\cdots = \frac{\pi-2}{4}$

Solution. Here $c=-\pi$ and $c+2l=\pi \Rightarrow l=\pi$

The Fourier series for $f(x)$ in $c\le x\le c+2l$ is given by

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right)+b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right) \quad \text{——— (i)}$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\,dx+\int_0^{\pi}\sin x\,dx\right]$$

$$= \frac{1}{\pi}\left[0+(-\cos x)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\cos\pi+\cos 0\right]$$

i.e., $a_0 = \frac{2}{\pi}$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (\because l=\pi)$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\cos nx\,dx + \int_0^{\pi}\sin x\cdot\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{2}\int_0^{\pi} 2\cos nx\sin x\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx \quad \left(\because 2\cos A\sin B = \sin(A+B)-\sin(A-B)\right)$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{\cos 0}{n+1} - \frac{\cos 0}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \quad \left(\because \cos(n\pm1)\pi = (-1)^{n\pm1}\right)$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\cancel{2}\pi}\left[(-1)^n\frac{(-\cancel{2})}{(n+1)(n-1)} + \frac{(-\cancel{2})}{(n+1)(n-1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n+1}}{n^2-1} - \frac{1}{n^2-1}\right]$$

i.e. $$a_n = \frac{\left[(-1)^{n+1}-1\right]}{\pi(n^2-1)} \quad (n\neq 1)$$

when $n=1$,

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\cos x\,dx + \int_0^{\pi}\sin x\cos x\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{2}\int_0^{\pi}(2\sin x\cos x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} \sin 2x\, dx\right]$$

$$= \frac{1}{2\pi}\left(-\frac{\cos 2x}{2}\right)_0^{\pi}$$

$$= -\frac{1}{4\pi}\left(\cos 2\pi - \cos 0\right)$$

i.e., $a_1 = -\frac{1}{4\pi}(1-1) = 0$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \quad (\because l = \pi)$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\sin nx\, dx + \int_0^{\pi}(\sin x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{2}\int_0^{\pi} 2\sin nx \sin x\, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\pi}{n+1}\right] - \frac{1}{2\pi}\left[\frac{\sin 0}{n-1} - \frac{\sin 0}{n+1}\right]$$

$$= \frac{1}{2\pi}(0) \quad (n \neq 1)$$

i.e., $b_n = 0 \quad (n \neq 1)$

when $n = 1$, $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\sin x\, dx + \int_0^{\pi}(\sin x)\sin x\, dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{2}\int_0^{\pi} 2\sin^2 x\, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}(1 - \cos 2x)\, dx$$

i.e., $b_1 = \frac{1}{2\pi}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi} = \frac{1}{2\pi}(\pi) = \frac{1}{2}$

Substituting a_0, a_n, b_n values in ①, we get

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x + \sum_{n=2}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{1}{2}\left(\frac{2}{\pi}\right) + (0)\cos x + \left(\frac{1}{2}\right)\sin x + \sum_{n=2}^{\infty}\left[\frac{[(-1)^{n+1}-1]}{\pi(n^2-1)}\cos nx + (0)\sin nx\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{[(-1)^{n+1}-1]}{n^2-1}\cos nx$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{1}{\pi}\left[\frac{(-1-1)}{2^2-1}\cos 2x + \frac{(1-1)}{3^2-1}\cos 3x + \frac{(-1-1)}{4^2-1}\cos 4x + \cdots\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{1}{\pi}\left[\frac{(-2)}{2^2-1}\cos 2x + \frac{(-2)}{4^2-1}\cos 4x + \frac{(-2)}{6^2-1}\cos 6x + \cdots\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n)^2-1}$$

i.e., $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$ —— ②

put $x = \frac{\pi}{2}$ in ②, we get

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2}\sin\frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos \cancel{2}n\left(\frac{\pi}{\cancel{2}}\right)}{4n^2-1}$$

i.e., $\sin\frac{\pi}{2} = \frac{1}{\pi} + \frac{1}{2}(1) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{4n^2-1}$

$$\Rightarrow 1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$

$$\Rightarrow -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = 1 - \frac{1}{2} - \frac{1}{\pi}$$

$$\Rightarrow -\frac{2}{\cancel{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)} = \frac{\pi-2}{2\cancel{\pi}}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)} = \frac{\pi-2}{4}$$

$$\Rightarrow \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots - \infty = \frac{\pi-2}{4}$$

Example 4. Obtain Fourier Series for the function

$$f(x) = \begin{cases} \pi x, & \underbrace{0}_{c} \le x \le 1 \\ \pi(2-x), & 1 \le x \le \underbrace{2}_{c+2l} \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$.

Solution: Here $c = 0$ and $c + 2l = 2 \Rightarrow l = 1$

The Fourier Series for the function $f(x)$ in $c \le x \le c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x) \quad \text{——— (1)}$$

where $a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\,dx$

$$= \frac{1}{1} \int_0^2 f(x)\,dx$$

$$= \int_0^1 (\pi x)\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi \int_0^1 x\,dx + \pi \int_1^2 (2-x)\,dx$$

$$= \pi \left(\frac{x^2}{2}\right)_0^1 + \pi \left(2x - \frac{x^2}{2}\right)_1^2$$

$$= \pi \left(\frac{1}{2} - 0\right) + \pi \left(4 - \frac{4}{2}\right) - \pi \left(2 - \frac{1}{2}\right)$$

$$= \frac{\pi}{2} + \frac{\cancel{4}^{\,2}\pi}{\cancel{2}} - \frac{3\pi}{2}$$

i.e., $a_0 = \pi$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{1} \int_0^2 f(x) \cos n\pi x \, dx \quad (\because l = 1)$$

$$= \int_0^1 (\pi x) \cos n\pi x \, dx + \int_1^2 \pi(2-x) \cos n\pi x \, dx$$

$$= \pi \int_0^1 x \cos n\pi x \, dx + \pi \int_1^2 (2-x) \cos n\pi x \quad [\because \text{By applying Leibnitz's Rule}]$$

$$= \pi \left[x \left(\frac{\sin n\pi x}{n\pi} \right) - (1) \left(-\frac{\cos n\pi x}{n^2\pi^2} \right) \right]_0^1 + \pi \left[(2-x) \left(\frac{\sin n\pi x}{n\pi} \right) - (0-1) \left(-\frac{\cos n\pi x}{n^2\pi^2} \right) \right]_1^2$$

$$= \pi \left[\left(\frac{\sin n\pi}{n\pi} + \frac{\cos n\pi}{n^2\pi^2} \right) - \left(0 + \frac{\cos 0}{n^2\pi^2} \right) \right] + \pi \left[\left(0 - \frac{\cos 2n\pi}{n^2\pi^2} \right) - (1) \left(\frac{\sin n\pi}{n\pi} \right) + \frac{\cos n\pi}{n^2\pi^2} \right]$$

$$= \pi \left[0 + \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} \right] + \pi \left[-\frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2} \right]$$

$$= 2\pi \left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} \right]$$

i.e., $a_n = \frac{2}{n^2\pi} \left[(-1)^n - 1 \right]$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin\left(\frac{n\pi x}{l} \right) dx$$

$$= \frac{1}{1} \int_0^2 f(x) \sin n\pi x \, dx \quad (\because l = 1)$$

$$= \int_0^1 (\pi x) \sin n\pi x \, dx + \int_1^2 \pi (2-x) \sin n\pi x \, dx$$

$$= \pi \int_0^1 x \sin n\pi x \, dx + \pi \int_1^2 (2-x) \sin n\pi x \, dx$$

$$= \pi \left[x \left(-\frac{\cos n\pi x}{n\pi} \right) - (1) \left(-\frac{\sin n\pi x}{n^2\pi^2} \right) \right]_0^1 + \pi \left[(2-x) \left(-\frac{\cos n\pi x}{n\pi} \right) - (0-1) \left(-\frac{\sin n\pi x}{n^2\pi^2} \right) \right]_1^2$$

$$= \pi \left[\left(-\frac{\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2} \right) - \left(0 + \frac{\sin 0}{n^2\pi^2} \right) \right] + \pi \left[\left(0 - \frac{\sin 0}{n^2\pi^2} \right) - \left(-\frac{\cos n\pi}{n\pi} - \frac{\sin n\pi}{n^2\pi^2} \right) \right]$$

$$= \pi \left[\left(-\frac{(-1)^n}{n\pi} + 0 \right) \right] + \pi \left[\frac{(-1)^n}{n\pi} \right]$$

i.e. $b_n = -\cancel{\frac{(-1)^n}{n}} + \cancel{\frac{(-1)^n}{n}} = 0$

substituting a_0, a_n, b_n values in ①, we get

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2}{n^2\pi} \left[(-1)^n - 1 \right] \cos n\pi x + (0) \sin n\pi x \right]$$

i.e., $f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos n\pi x$ —— (2)

Put $x = 2$ in (2), we get

$$f(2) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \underbrace{\cos 2n\pi}_{1}$$

i.e., $0 = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2}$ $\left[\because f(x) = \pi(2-x),\ 1 \le x \le 2 \right.$ So, $\left. f(2) = \pi(2-2) = 0 \right]$

$$\Rightarrow \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} = -\frac{\pi}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} = -\frac{\pi^2}{4}$$

$$\Rightarrow \frac{(-1-1)}{1^2} + \frac{(1-1)}{2^2} + \frac{(-1-1)}{3^2} + \frac{(1-1)}{4^2} + \frac{(-1-1)}{5^2} + \cdots = -\frac{\pi^2}{4}$$

$$\Rightarrow \frac{(-2)}{1^2} + \frac{(-2)}{3^2} + \frac{(-2)}{5^2} + \cdots = -\frac{\pi^2}{4}$$

$$\Rightarrow \cancel{+}2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \cancel{+}\frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$$

HW

Example 5. Find the Fourier Series for $f(x) = \begin{cases} x, & \overset{c}{0} \le x \le 3 \\ 6-x, & 3 \le x \le \underset{c+2l}{6} \end{cases}$

Solution: Here $c = 0$ & $c + 2l = 6 \Rightarrow l = 3$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{3}\right) + b_n \sin\left(\frac{n\pi x}{3}\right) \right]$ —— (1)

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\,dx = \frac{1}{3} \int_0^6 f(x)\,dx = \frac{1}{3}\left[\int_0^3 x\,dx + \int_3^6 (6-x)\,dx\right] = 3$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx = \frac{1}{3} \int_0^6 f(x) \cos\left(\frac{n\pi x}{3}\right) dx = \frac{6}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{1}{3} \int_0^6 f(x) \sin\left(\frac{n\pi x}{3}\right) dx = 0$$

Substituting a_0, a_n, b_n values in (1), we obtain

$$f(x) = \frac{3}{2} + \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos\left(\frac{n\pi x}{3}\right)$$

Even or odd functions

Even and Odd functions:

①

→ A function $f(x)$ is said to be even in $[-l, l]$ if $f(-x) = f(x)\ \forall x \in [-l, l]$

e.g: x^2, $\cos x$, $x \sin x$, $|x|$, $|\sin x|$, $|\cos x|$, ... etc. are all even functions

→ A function $f(x)$ is said to be odd in $[-l, l]$ if $f(-x) = -f(x)\ \forall x \in [-l, l]$

e.g: x, $\sin x$, $x\cos x$, x^3, ... etc. are all odd functions

Fourier Series for even and odd functions:

The Fourier series for $f(x)$ in the interval $-l \le x \le l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right] \quad \text{—— ①}$$

wher $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

When $f(x)$ is even function: $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_{0}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)\cos\left(\frac{n\pi x}{l}\right)}_{\text{even function}}dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)\sin\left(\frac{n\pi x}{l}\right)}_{\text{odd function}}dx = \frac{1}{l}(0) = 0$$

Thus, if a periodic function $f(x)$ is even, its Fourier Series expansion contains only cosine terms.

When $f(x)$ is odd function: $a_0 = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{\text{odd function}}dx = \frac{1}{l}(0) = 0$

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)\cos\left(\frac{n\pi x}{l}\right)}_{\text{odd function}}dx = \frac{1}{l}(0) = 0$$

②

$$b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)\sin\left(\frac{n\pi x}{l}\right)}_{\text{even function}} dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

Thus, if a periodic function $f(x)$ is odd, its Fourier Series expansion contains only sine terms.

Formulae: 1) If $f(x)$ is an even function in $-l \le x \le l$ then its Fourier Series expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

2) If $f(x)$ is on odd function in $-l \le x \le l$ then its Fourier Series expansion is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$

Example 1. Express $f(x) = \frac{x}{2}$ as a Fourier series in the interval $-\pi < x < \pi$

Solution: since $f(-x) = -f(x)$, $f(x)$ is an odd function in $(-\pi, \pi)$

Here $l = \pi$ $\left[\because (-l, l) = (-\pi, \pi)\right]$

The Fourier series for an odd function $f(x)$ in $-l < x < l$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——} ① \quad (\because l = \pi)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\cdot \sin nx\, dx \quad (\because l = \pi)$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\pi\left(-\frac{\cos n\pi}{n}\right) + \frac{\sin n\pi}{n^2} + 0\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(-1)^n + 0\right] \quad (\because \sin n\pi = 0)$$

i.e., $b_n = \dfrac{(-1)^{n+1}}{n}$

Substituting b_n value in ①, we get

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

Example 2. Find a Fourier Series to represent x^2 in the interval $(-l, l)$

Solution: Let $f(x) = x^2$

Since $f(-x) = f(x)$, $f(x)$ is an even function in $(-l, l)$

The Fourier Series for an even function $f(x)$ in $(-l, l)$ is

given by $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\dfrac{n\pi x}{l}\right)$ —— ①

where $a_0 = \dfrac{2}{l}\int_0^l f(x)\,dx = \dfrac{2}{l}\int_0^l x^2\,dx = \dfrac{2}{l}\left(\dfrac{x^3}{3}\right)_0^l = \dfrac{2l^2}{3}$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l \underset{u}{x^2} \underset{v}{\cos\left(\frac{n\pi x}{l}\right)}dx \quad [\because \text{By Leibnitz's Rule}]$$

$$= \frac{2}{l}\left[x^2\left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - (2x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right) + (2)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[l^2\left(\frac{l}{n\pi}\cdot \underset{0}{\sin n\pi}\right) + 2l\left(\frac{l^2}{n^2\pi^2}\cdot\cos n\pi\right) - 2\frac{l^3}{n^3\pi^3}\underset{0}{\sin n\pi}\right]$$

$$- \frac{2}{l}[0]$$

$$= \frac{2}{l}\left[\frac{2l^3}{n^2\pi^2}(-1)^n\right] \quad (\because \cos n\pi = (-1)^n)$$

i.e., $a_n = \frac{4l^2}{n^2\pi^2}(-1)^n$

Substituting a_0, a_n values in ①, we get

$$x^2 = \frac{1}{2}\left(\frac{2l^2}{3}\right) + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}(-1)^n \cos\left(\frac{n\pi x}{l}\right)$$

i.e., $$x^2 = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

Example 3. Obtain a Fourier series for $f(x) = |x|, -\pi < x < \pi$.

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$.

Solution. Since $f(-x) = |-x| = |x| = f(x)$, $f(x)$ is an even function

The Fourier series for $f(x)$ in $-l < x < l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——— ①} \quad (\because l = \pi)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{\pi}\int_0^\pi |x|\,dx \qquad (\because l = \pi)$$

$$= \frac{2}{\pi}\int_0^\pi x\,dx \qquad (\because |x| = x \text{ for } 0 < x < \pi)$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^\pi$$

$$= \pi$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{\pi}\int_0^\pi |x| \cos\left(\frac{n\pi x}{\pi}\right)dx \qquad (\because l = \pi)$$

$$= \frac{2}{\pi}\int_0^\pi \underset{u}{x}\ \underset{v}{\cos nx}\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\pi\left(\frac{\sin n\pi}{n}\right) + \frac{\cos n\pi}{n^2}\right] - \frac{2}{\pi}\left[0 + \frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}(0) + \frac{(-1)^n}{n^2}\right] - \frac{2}{\pi}\left[\frac{1}{n^2}\right]$$

i.e., $a_n = \frac{2}{\pi n^2}[(-1)^n - 1]$

Substituting a_0, a_n values in ①, we get

$$|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[(-1)^n - 1]\cos nx$$

i.e., $|x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2}\cos nx$ ——— ②

Deduction: put $x = 0$ in ②, we get

$$|0| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2}\cos 0$$

$$\Rightarrow \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} = -\frac{\pi}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} = -\frac{\pi^2}{4}$$

$$\Rightarrow \frac{(-1-1)}{1^2} + \frac{(1-1)}{2^2} + \frac{(-1-1)}{3^2} + \frac{(1-1)}{4^2} + \frac{(-1-1)}{5^2} + \cdots = -\frac{\pi^2}{4}$$

$$\Rightarrow \frac{(-2)}{1^2} + \frac{(-2)}{3^2} + \frac{(-2)}{5^2} + \cdots = -\frac{\pi^2}{4}$$

$$\Rightarrow \cancel{-}2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \cancel{-}\frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Example 4. Expand $f(x) = |\cos x|$ as a Fourier Series in the interval $(-\pi, \pi)$

Solution. Since $f(-x) = |\cos(-x)| = |\cos x| = f(x)$, $f(x)$ is an even function

The Fourier series for $f(x)$ in $(-l, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ ——— ① } (\because l = \pi$$

Where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ ⑥

$= \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx \quad (\because l=\pi)$

$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\,dx + \int_{\pi/2}^{\pi}(-\cos x)\,dx\right]$

$$\left[\because |\cos x| = \begin{cases} \cos x, & 0<x<\frac{\pi}{2} \quad \text{Q-I} \\ -\cos x, & \frac{\pi}{2}<x<\pi \quad \text{Q-II}\end{cases}\right]$$

$= \frac{2}{\pi}\left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right]$

$= \frac{2}{\pi}\left[(\sin\frac{\pi}{2} - \sin 0) - (\sin\pi - \sin\frac{\pi}{2})\right]$

i.e., $a_0 = \frac{4}{\pi}$

$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$= \frac{2}{\pi}\int_0^{\pi}|\cos x|\cos nx\,dx \quad (\because l=\pi)$

$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cos nx\,dx + \int_{\pi/2}^{\pi}(-\cos x)\cos nx\,dx\right]$

$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos nx\cos x\,dx - \int_{\pi/2}^{\pi} 2\cos nx\cos x\,dx\right]$

$= \frac{1}{\pi}\left[\int_0^{\pi/2}\{\cos(n+1)x + \cos(n-1)x\}dx - \int_{\pi/2}^{\pi}\{\cos(n+1)x + \cos(n-1)x\}dx\right]$

$= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}$

$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} - 0\right] - \frac{1}{\pi}\left[\frac{\sin(n+1)\pi}{n+1} + \frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\frac{\pi}{2}}{n+1} - \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right]$

$= \frac{1}{\pi}\left[\frac{\sin(\frac{\pi}{2}+\frac{n\pi}{2})}{n+1} - \frac{\sin(\frac{\pi}{2}-\frac{n\pi}{2})}{n-1}\right] - \frac{1}{\pi}\left[0 - \frac{\sin(\frac{\pi}{2}+\frac{n\pi}{2})}{n+1} + \frac{\sin(\frac{\pi}{2}-\frac{n\pi}{2})}{n-1}\right]$

$$= \frac{2}{\pi}\left[\frac{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)}{n+1} - \frac{\sin\left(\frac{\pi}{2}-\frac{n\pi}{2}\right)}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right] \quad \left(\because \sin\left(\frac{\pi}{2}\pm\theta\right) = \cos\theta\right)$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]\cos\frac{n\pi}{2}$$

$$= \frac{2}{\pi}\left[\frac{\cancel{n}-1-\cancel{n}-1}{(n+1)(n-1)}\right]\cos\frac{n\pi}{2}$$

i.e., $a_n = \dfrac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)} \quad (n \neq 1)$

When $n=1$, $a_1 = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos x \, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cdot \cos x \, dx + \int_{\pi/2}^{\pi} (-\cos x)\cos x \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos^2 x \, dx - \int_{\pi/2}^{\pi} 2\cos^2 x \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} (1+\cos 2x)\,dx - \int_{\pi/2}^{\pi} (1+\cos 2x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(x+\frac{\sin 2x}{2}\right)_0^{\pi/2} - \left(x+\frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right) - 0 - \left(\pi+\frac{\sin 2\pi}{2}\right) + \left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \frac{\pi}{2}\right] \quad (\because \sin\pi = \sin 2\pi = 0)$$

i.e., $a_1 = \frac{1}{\pi}(\pi - \pi) = 0$

Substituting a_0, a_n values in ①, we get

$$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

i.e., $|\cos x| = \frac{1}{2}\left(\frac{4}{\pi}\right) + (0)\cos x + \displaystyle\sum_{n=2}^{\infty} \frac{-4\cos\frac{n\pi}{2}}{\pi(n^2-1)}\cos nx$

or $|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty} \frac{\cos\frac{n\pi}{2}}{n^2-1}\cos nx$

or $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} - \cdots\right)$

Example 5. Find the Fourier Series to represent the function $f(x) = |\sin x|,\ -\pi < x < \pi$.

Solution: Since $f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$, $f(x)$ is an even function.

The Fourier Series for $f(x)$ in $-l < x < l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)} \quad (\because l = \pi)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x|\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\,dx \quad \left(\because |\sin x| = \sin x \text{ for } 0 < x < \pi\right)$$

$$= \frac{4}{\pi}$$

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos nx \sin x\,dx$$

$$= -\frac{2}{\pi(n^2-1)}\left[(-1)^n + 1\right] \quad (n \neq 1)$$

when $n = 1$, $a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\,dx = 0$

Substituting a_0, a_n values in (1), we get

$$|\sin x| = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{\left[(-1)^n + 1\right]}{n^2-1}\cos nx$$

Example 6. Obtain the Fourier series for the function $f(x)$ given by

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Solution: We have $f(-x) = \begin{cases} 1 - \frac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1 + \frac{2x}{\pi}, & 0 \le -x \le \pi \end{cases}$

$$= \begin{cases} 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \\ 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \end{cases}$$

$$= f(x)$$

Since $f(-x) = f(x)$, $f(x)$ is an even function in $-\pi \le x \le \pi$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—— (1)} \quad (\because l = \pi)$$

where $a_0 = \frac{2}{l} \int_0^l f(x)\,dx$

$$= \frac{2}{\pi} \int_0^{\pi} f(x)\,dx \quad (\because l = \pi)$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \left(x - \frac{2}{\pi} \cdot \frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{2}{\pi} \left(\pi - \frac{\pi^2}{\pi}\right) - 0$$

i.e., $a_0 = 0$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \quad (\because l = \pi)$$

$$= \frac{2}{\pi} \int_0^{\pi} \Big(\underbrace{1 - \frac{2x}{\pi}}_{u}\Big) \underbrace{\cos nx}_{v} \, dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(0-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2\pi}{\pi}\right)\left(\frac{\sin n\pi}{n}\right) - \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2}\right) - 0 + \frac{2}{\pi}\cdot\left(\frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[(1-2)(0) - \frac{2}{\pi n^2}(-1)^n + \frac{2}{\pi n^2}\right]$$

i.e., $a_n = \dfrac{4[1-(-1)^n]}{n^2\pi^2}$

Substituting a_0, a_n values in ①, we get

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \frac{4[1-(-1)^n]}{n^2\pi^2}\cos nx$$

i.e., $f(x) = \dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos nx$ —— ②

put $x=0$ in ②, we get

$$f(0) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos 0$$

i.e., $1+\dfrac{2(0)}{\pi} = \dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}$ $\left[\because f(x) = 1+\frac{2x}{\pi}, -\pi \le x \le 0\right]$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2} = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{(1+1)}{1^2} + \frac{(1-1)}{2^2} + \frac{(1+1)}{3^2} + \frac{(1-1)}{4^2} + \cdots = \frac{\pi^2}{4}$$

$$\Rightarrow 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right) = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$$

HW

Example 7. Given $f(x) = \begin{cases} -x+1 & \text{for } -\pi \le x \le 0, \\ x+1 & \text{for } 0 \le x \le \pi. \end{cases}$ Is the function $f(x)$ even or odd? Find the Fourier series for $f(x)$ and deduce the value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

Solution: Since $f(-x) = f(x)$, $f(x)$ is even function

$a_0 = \pi+2$ and $a_n = \dfrac{2}{n^2\pi}[(-1)^n - 1]$

$$\therefore f(x) = \frac{1}{2}(\pi+2) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]}{n^2}\cos nx$$

Half range Fourier series

Half-Range Fourier Series:

Sometimes it is required to expand $f(x)$ as a Fourier Series in the half-range $(0, l)$ but not in the full range $(-l, l)$. Such a Series is known as half-range Fourier series.

→ Half-range Fourier sine series: The half-range Fourier sine series for $f(x)$ in $0 < x < l$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

→ Half-range Fourier cosine series: The half-rang Fourier cosine Series for $f(x)$ in $0 < x < l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x) dx$

$a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

Example 1. Express $f(x) = x$ as a half-range sine series in $0 < x < 2$

solution. The half-range Fourier sine Series for $f(x)$ in $0 < x < l$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \quad (\because l = 2) \quad \text{—— (1)}$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$$= \frac{2}{2}\int_0^2 \underset{u}{x} \, \underset{v}{\sin\left(\frac{n\pi x}{2}\right)} dx \quad (\because l = 2)$$

$$= \left[x\left(-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right) - (1)\left(-\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_0^2$$

$$= \left[2\left(-\frac{2}{n\pi}\cdot\cos n\pi\right) + \left(\frac{4}{n^2\pi^2}\sin n\pi\right) + 0\right] \quad (\because \sin n\pi = 0)$$

i.e., $b_n = -\frac{4}{n\pi}(-1)^n = \frac{4}{n\pi}(-1)^{n+1}$

Substituting the value of b_n in ①, we get

$$x = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{2}\right)$$

Example 2. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0<x<1$. Hence show that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}$$

Solution: Given $f(x) = (x-1)^2$.

Here $(0, l) = (0, 1) \Rightarrow l = 1$

The half-range Fourier cosine series for $f(x)$ in $0<x<l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \text{——— ①} \quad (\because l = 1)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{1}\int_0^1 (x-1)^2\,dx$$

$$= 2\int_0^1 (x^2-2x+1)\,dx$$

$$= 2\left[\frac{x^3}{3} - x^2 + x\right]_0^1$$

i.e., $a_0 = \frac{2}{3}$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 (x-1)^2 \cos n\pi x\, dx \quad (\because l=1)$$

$$= 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[\left(0 - 0 - 2\frac{\sin n\pi}{n^3\pi^3}\right) - \left(0 - 2\frac{\cos 0}{n^2\pi^2} - 0\right)\right]$$

$$= 2\left[\frac{2}{n^2\pi^2}\right] \quad (\because \sin n\pi = 0)$$

i.e., $a_n = \dfrac{4}{n^2\pi^2}$

Substituting a_0, a_n values in ①, we get

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2} \quad \text{——②}$$

put $x=0$ in ②, we get

$$(0-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 0}{n^2}$$

$$\Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 - \frac{1}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty = \frac{\pi^2}{6}$$

Example 3. Obtain the Fourier expansion of $x\sin x$ as a cosine series in $(0, \pi)$. Hence show that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots \infty = \frac{\pi - 2}{4}$.

Solution: ~~Let~~ $f(x) = x\sin x$

Here $(0, l) = (0, \pi) \Rightarrow l = \pi$

The half-range Fourier cosine series for $f(x)$ in $(0, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—— (1)} \quad (\because l = \pi)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} \underset{u}{x}\,\underset{v}{\sin x}\,dx$$

$$= \frac{2}{\pi}\Big[x(-\cos x) - (1)(-\sin x)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\Big[(-\pi\cos\pi + \sin\pi) - (0)\Big]$$

i.e., $a_0 = \frac{2}{\pi}(\pi) = 2$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(x\sin x)\cos nx\,dx \quad (\because l = \pi)$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,[2\cos nx \sin x]\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \underset{u}{x}\,[\underbrace{\sin(n+1)x - \sin(n-1)x}_{v}]\,dx$$

[$\because$ By Leibnitz's Rule]

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{-\frac{\sin(n+1)\pi}{(n+1)^2} + \frac{\sin(n-1)\pi}{(n-1)^2}\right\}\right]$$

$$-\frac{1}{\pi}[0] \quad (n \neq 1)$$

$$= \left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right] \quad \left(\because \cos(n\pm1)\pi = (-1)^{n\pm1} \text{ and } \sin(n\pm1)\pi = 0\right)$$

$$= \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1}$$

$$= \left[\frac{1}{n+1} - \frac{1}{n-1}\right](-1)^n$$

$$= \frac{(n-1-n-1)}{(n+1)(n-1)}(-1)^n$$

i.e., $a_n = \dfrac{2(-1)^{n+1}}{n^2-1} \quad (n \neq 1)$

when $n=1$, $\quad a_1 = \dfrac{2}{\pi}\displaystyle\int_0^\pi (x\sin x)\cos x\, dx$

$$= \frac{1}{\pi}\int_0^\pi x\,(2\sin x\cos x)\,dx$$

$$= \frac{1}{\pi}\int_0^\pi \underset{u}{\underset{\downarrow}{x}}\ \underset{v}{\underset{\downarrow}{\sin 2x}}\, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi}{2}\cos 2\pi + \frac{\sin 2\pi}{4}\right) - 0\right]$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{2}(1) + 0\right) \quad (\because \cos 2\pi = 1 \ \&\ \sin 2\pi = 0)$$

i.e., $a_1 = -\dfrac{1}{2}$

Substituting a_0, a_n values in ①, we get

$$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

i.e., $x\sin x = \dfrac{1}{2}(2) + \left(-\dfrac{1}{2}\right)\cos x + \displaystyle\sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx$

or $\quad x\sin x = 1 - \dfrac{1}{2}\cos x + 2\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nx$ —— ②

Put $x = \frac{\pi}{2}$ in ②, we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

i.e., $\quad \dfrac{\pi}{2} = 1 - \dfrac{1}{2}(0) + 2\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$

$$\Rightarrow 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2} = \frac{\pi}{2}-1$$

$$\Rightarrow 2\left[\frac{(-1)}{2^2-1}\cdot\cos\pi + \frac{1}{3^2-1}\cos\frac{3\pi}{2} + \frac{(-1)}{4^2-1}\cos 2\pi + \cdots\right] = \frac{\pi-2}{2}$$

$$\Rightarrow 2\left[\frac{1}{2^2-1} + \frac{1}{3^2-1}(0) - \frac{1}{4^2-1} + \frac{1}{5^2-1}(0) + \frac{1}{6^2-1} - \cdots\right] = \frac{\pi-2}{2}$$

$$\Rightarrow \frac{1}{(2-1)(2+1)} - \frac{1}{(4-1)(4+1)} + \frac{1}{(6-1)(6+1)} - \cdots\infty = \frac{\pi-2}{4}$$

$$\Rightarrow \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\infty = \frac{\pi-2}{4}$$

Example 4. Expand $f(x) = \begin{cases} \frac{1}{4}-x, & 0<x<\frac{1}{2} \\ x-\frac{3}{4}, & \frac{1}{2}<x<1 \end{cases}$ as a Fourier series of sine terms.

Solution: Here $(0,l) = (0,1) \Rightarrow l=1$

The half-range Fourier sine series for $f(x)$ in $0<x<l$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{—— (1)} \quad (\because l=1)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx \quad (\because l=1)$$

$$= 2\left[\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin n\pi x\,dx + \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin n\pi x\,dx\right]$$

$$= 2\left[\left(\frac{1}{4}-x\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (0-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^{1/2}$$

$$+ 2\left[\left(x-\frac{3}{4}\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1-0)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{1/2}^{1}$$

$$= 2\left[\left\{\left(\frac{1}{4}-\frac{1}{2}\right)\left(\frac{-\cos\frac{n\pi}{2}}{n\pi}\right) - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right\} - \left\{\left(\frac{1}{4}-0\right)\left(-\frac{\cos 0}{n\pi}\right) - \frac{\sin 0}{n^2\pi^2}\right\}\right]$$

$$+ 2\left[\left\{(1-\tfrac{3}{4})\left(-\frac{\cos n\pi}{n\pi}\right) + \frac{\sin n\pi}{n^2\pi^2}\right\} - \left\{\left(\frac{1}{2}-\frac{3}{4}\right)\left(\frac{-\cos\frac{n\pi}{2}}{n\pi}\right) + \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right\}\right]$$

$$= 2\left[\cancel{\frac{1}{4n\pi}\cos\frac{n\pi}{2}} - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{1}{4n\pi} - \frac{1}{4n\pi}\cos n\pi - 0 - \cancel{\frac{1}{4n\pi}\cos\frac{n\pi}{2}} - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

i.e., $b_n = 2\left[\frac{1}{4n\pi} - \frac{(-1)^n}{4n\pi} - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = 2\left[\frac{1-(-1)^n}{4n\pi} - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$

Substituting b_n value in ①, we get

$$f(x) = 2\sum_{n=1}^{\infty}\left\{\frac{[1-(-1)^n]}{4n\pi} - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\sin n\pi x$$

Example 5. Find the Fourier cosine series of $f(x) = \pi - x$ in $0 < x < \pi$. Hence show that $\sum_{r=0}^{\infty}\frac{1}{(2r+1)^2} = \frac{\pi^2}{8}$.

Solution. Given $f(x) = \pi - x$

Here $(0, l) = (0, \pi) \Rightarrow l = \pi$

The half-range Fourier cosine series for $f(x)$ in $0 < x < \pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——} ① \quad (\because l = \pi)$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi n^2}[1-(-1)^n]$$

Substituting a_0, a_n values in ①, we get

$$\pi - x = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos nx \quad \text{——} ②$$

Put $x=0$ in ②, we get

$$\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2} = \frac{\pi^2}{4}$$

i.e., $\frac{2}{2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots = \frac{\pi^2}{4}$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

$$\Rightarrow \sum_{r=0}^{\infty}\frac{1}{(2r+1)^2} = \frac{\pi^2}{8}$$

Example 6. Find the half-range sine series for $f(x) = x\cos x$ in $(0,\pi)$

Solution: The half-range Fourier sine series for $f(x)$ in $(0,\pi)$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ —— ① $(\because l=\pi)$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi} x(2\sin nx\cos x)\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x + \sin(n-1)x]\,dx$$

i.e., $b_n = \frac{2(-1)^n \cdot n}{n^2-1}$ $(n \neq 1)$

when $n=1$, $b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx = -\frac{1}{2}$

Substituting the value of b_n in ①, we get

$$f(x) = b_1\sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

i.e., $x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^n \cdot n}{n^2-1}\sin nx$

Fourier transforms

Fourier Transforms

Introduction: Fourier Transform is a mathematical procedure which transforms a function from time domain to frequency domain. Fourier Transform is very useful in many areas of engineering such as circuit analysis, signal processing, Signal analysis, image processing & filtering. It is also used to solve Initial Boundary Value Problems (IBVPs) in the fields of conduction of heat, free and forced vibrations of a membrane, transmission lines, etc.

Fourier Transform: The Fourier transform of $f(x)$, $-\infty < x < \infty$ is denoted by $F\{f(x)\}$ or $F(p)$, and is defined as

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{ipx} dx = F(p)$$

Inversion formula for Fourier tansform: If $F(p)$ is the Fourier transform of $f(x)$ then $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p)\, e^{-ipx} dp$

Fourier sine and cosine transforms: The Fourier sine transform of $f(x)$, $0 < x < \infty$ is denoted by $F_s\{f(x)\}$ or $F_s(p)$, and is defined as $F_s\{f(x)\} = \int_0^{\infty} f(x) \sin px\, dx = F_s(p)$

→ The Fourier cosine transform of $f(x)$, $0 < x < \infty$ is denoted by $F_c\{f(x)\}$ or $F_c(p)$, and is defined as

$$F_c\{f(x)\} = \int_0^{\infty} f(x) \cos px\, dx = F_c(p)$$

Inversion formula for Fourier sine and cosine transforms:

a) If $F_s(p)$ is the Fourier sine transform of $f(x)$ then

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(p)\sin px\,dp$$

b) If $F_c(p)$ is the Fourier cosine transform of $f(x)$ then

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(p)\cos px\,dp$$

Example 1. Find the Fourier transform of $f(x) = \begin{cases} 1, & |x|<a \\ 0, & |x|>a. \end{cases}$

Hence evaluate $\int_0^\infty \frac{\sin ax}{x}dx$

Solution. Given $f(x) = \begin{cases} 1, & |x|<a \text{ i.e., } -a<x<a \\ 0, & |x|>a \text{ i.e., } x<-a \text{ or } x>a \end{cases}$

The Fourier transform of $f(x)$ is given by

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\,e^{ipx}dx$$

$$= \int_{-\infty}^{-a}(0)e^{ipx}dx + \int_{-a}^{a}(1)e^{ipx}dx + \int_{a}^{\infty}(0)e^{ipx}dx$$

$$= \int_{-a}^{a} e^{ipx}dx$$

$$= \int_{-a}^{a}(\cos px + i\sin px)dx \quad (\because e^{ipx} = \cos px + i\sin px)$$

$$= \int_{-a}^{a}\underbrace{\cos px}_{\text{even } f^n}dx + i\int_{-a}^{a}\underbrace{\sin px}_{\text{odd } f^n}dx$$

$$= 2\int_0^a \cos px\,dx + i(0)$$

$$= 2\left(\frac{\sin px}{p}\right)_0^a$$

i.e., $F\{f(x)\} = \frac{2}{p}(\sin pa - \sin 0) = \frac{2\sin pa}{p} = F(p)$, say

By inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\, e^{-ipx}\, dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin pa}{p}\, e^{-ipx}\, dp$$

i.e., $$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin pa}{p}\, e^{-ipx}\, dp$$

put $x = 0$, we get

$$f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin pa}{p}\, e^{-ip(0)}\, dp$$

i.e., $$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ap}{p}\, dp \qquad (\because f(x) = 1 \text{ for } -a < x < a)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin ap}{p}\, dp = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin ap}{p}\, dp = \pi \qquad \left(\because \frac{\sin ap}{p} \text{ is an even function of } p\right)$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin ap}{p}\, dp = \frac{\pi}{2}$$

changing p to x, we get

$$\int_{0}^{\infty} \frac{\sin ax}{x}\, dx = \frac{\pi}{2}$$

Example 2. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

Hence evaluate $\int_{0}^{\infty} \frac{(x\cos x - \sin x)}{x^3} \cos\left(\frac{x}{2}\right) dx$.

Solution. The Fourier transform of $f(x)$ is given by

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx$$

$$= \int_{-\infty}^{-1} (0)\, e^{ipx}\, dx + \int_{-1}^{1} (1-x^2)\, e^{ipx}\, dx + \int_{1}^{\infty} (0)\, e^{ipx}\, dx$$

$$= \int_{-1}^{1} (1-x^2)\,(\cos px + i\sin px)\,dx$$

$$= \int_{-1}^{1} \underbrace{(1-x^2)\cos px}_{\text{even } f^n}\,dx + i\int_{-1}^{1} \underbrace{(1-x^2)\sin px}_{\text{odd } f^n}\,dx$$

$$= 2\int_{0}^{1} (\underset{u}{1-x^2})\,\underset{v}{\cos px}\,dx + i(0)$$

$$= 2\left[(1-x^2)\left(\frac{\sin px}{p}\right) - (-2x)\left(-\frac{\cos px}{p^2}\right) + (-2)\left(-\frac{\sin px}{p^3}\right)\right]_0^1$$

$$= 2\left[0 - \frac{2}{p^2}\cos p + \frac{2}{p^3}\sin p\right] - 2\,[0]$$

i.e., $F\{f(x)\} = \frac{4}{p^3}(\sin p - p\cos p) = F(p)$, say

By inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\,e^{-ipx}\,dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{p^3}(\sin p - p\cos p)\,e^{-ipx}\,dp$$

i.e., $f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(\sin p - p\cos p)}{p^3}\,e^{-ipx}\,dp$

Put $x = \frac{1}{2}$, we get

$$f\left(\tfrac{1}{2}\right) = \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin p - p\cos p}{p^3}\right) e^{-i\frac{p}{2}}\,dp \quad \left[\because f(x) = 1-x^2,\ -1 \le x \le 1\right]$$

i.e., $1 - \left(\frac{1}{2}\right)^2 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(\sin p - p\cos p)}{p^3}\left(\cos\frac{p}{2} - i\sin\frac{p}{2}\right) dp$

$$\Rightarrow \frac{\pi}{2}\left(\frac{3}{4}\right) = \int_{-\infty}^{\infty} \underbrace{\frac{(\sin p - p\cos p)}{p^3}\cos\frac{p}{2}}_{\text{even } f^n}\,dp - i\int_{-\infty}^{\infty} \underbrace{\frac{(\sin p - p\cos p)}{p^3}\sin\frac{p}{2}}_{\text{odd } f^n}\,dp$$

$$\Rightarrow \frac{3\pi}{3} = 2\int_{0}^{\infty} \frac{(\sin p - p\cos p)}{p^3}\cos\frac{p}{2}\,dp - i(0)$$

$$\Rightarrow \int_0^\infty \frac{(\sin p - p\cos p)}{p^3}\cos\left(\frac{p}{2}\right)dp = \frac{3\pi}{16}$$

$$\Rightarrow \int_0^\infty \frac{(p\cos p - \sin p)}{p^3}\cos\left(\frac{p}{2}\right)dp = -\frac{3\pi}{16}$$

Example 3. Find the Fourier transform of $e^{-a^2x^2}$ $(a>0)$. Hence deduce that $e^{-x^2/2}$ is self reciprocal in respect of Fourier transform.

Solution. Let $f(x) = e^{-a^2x^2}$

By definition, $F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{ipx} dx$

$$\therefore F\{e^{-a^2x^2}\} = \int_{-\infty}^{\infty} e^{-a^2x^2} \cdot e^{ipx} dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2 + ipx} dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x^2 - \frac{ip}{a^2}x\right)} dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left\{x^2 - 2\cdot x\cdot\left(\frac{ip}{2a^2}\right) + \left(\frac{ip}{2a^2}\right)^2 - \left(\frac{ip}{2a^2}\right)^2\right\}} dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left\{\left(x - \frac{ip}{2a^2}\right)^2 + \frac{p^2}{4a^4}\right\}} dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x - \frac{ip}{2a^2}\right)^2 - \frac{p^2}{4a^2}} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[a\left(x - \frac{ip}{2a^2}\right)\right]^2} \cdot e^{-\frac{p^2}{4a^2}} dx$$

put $y = a\left(x - \frac{ip}{2a^2}\right)$ so that $dy = a\,dx$

$$= \int_{-\infty}^{\infty} e^{-y^2} \cdot e^{-\frac{p^2}{4a^2}} \cdot \frac{dy}{a}$$

$$= \frac{1}{a} e^{-\frac{p^2}{4a^2}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{a} e^{-\frac{p^2}{4a^2}} \cdot (\sqrt{\pi}) \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}\right]$$

i.e., $F\{e^{-a^2x^2}\} = \frac{\sqrt{\pi}}{a} \cdot e^{-\frac{p^2}{4a^2}}$ ——— (1)

Put $a^2 = \frac{1}{2}$ so that $a = \frac{1}{\sqrt{2}}$ in (1), we get

$$F\{e^{-x^2/2}\} = \frac{\sqrt{\pi}}{\frac{1}{\sqrt{2}}} \cdot e^{\frac{-p^2}{4(\frac{1}{2})}}$$

$$= \sqrt{2\pi} \cdot e^{-p^2/2}$$

i.e., Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-p^2/2}$. Also $e^{-x^2/2}$ and $e^{-p^2/2}$ are the same. Hence it follows that $e^{-x^2/2}$ is self-reciprocal under the Fourier transform.

Example 4. Find the Fourier cosine transform of $f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

Solution. By the definition of Fourier cosine transform, $F_c\{f(x)\} = \int_0^\infty f(x) \cos px \, dx$

$$= \int_0^1 \underset{u}{(x)} \underset{v}{\cos px} \, dx + \int_1^2 \underset{u}{(2-x)} \underset{v}{\cos px} \, dx$$

$$+ \int_2^\infty (0) \cos px \, dx$$

$$= \left[x\left(\frac{\sin px}{p}\right) - (1)\left(-\frac{\cos px}{p^2}\right)\right]_0^1$$

$$+ \left[(2-x)\left(\frac{\sin px}{p}\right) - (-1)\left(-\frac{\cos px}{p^2}\right)\right]_1^2$$

$$= \left[\left(\cancel{\frac{\sin p}{p}} + \frac{\cos p}{p^2} \right) - \left(0 + \frac{\cos 0}{p^2} \right) \right] + \left[\left(0 - \frac{\cos 2p}{p^2} \right) - \left(\cancel{\frac{\sin p}{p}} - \frac{\cos p}{p^2} \right) \right]$$

$$= \frac{\cos p}{p^2} - \frac{1}{p^2} - \frac{\cos 2p}{p^2} + \frac{\cos p}{p^2}$$

i.e, $F_c\{f(x)\} = \frac{1}{p^2}(2\cos p - \cos 2p - 1)$

Example 5. Find the Fourier sine transform of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$, $m>0$

Solution. Let $f(x) = e^{-|x|}$.

By the definition of Fourier sine transform,

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin px \, dx$$

$$= \int_0^\infty e^{-x} \sin px \, dx \qquad (\because |x| = x \text{ for } 0 < x < \infty)$$

$$= \left[\frac{e^{-x}}{(-1)^2 + p^2} (-\sin px - p\cos px) \right]_{x=0}^{\infty}$$

$$= \left[0 - \frac{e^{-0}}{p^2+1} (-\sin 0 - p\cos 0) \right]$$

i.e., $F_s\{f(x)\} = \frac{p}{p^2+1} = F_s(p)$, say

By inversion formula for Fourier sine transform,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(p) \sin px \, dp$$

i.e., $e^{-|x|} = \frac{2}{\pi} \int_0^\infty \frac{p}{p^2+1} \sin px \, dp$

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or $\int_0^\infty \frac{p\sin px}{p^2+1}\,dp = \frac{\pi}{2} e^{-|x|}$

Put $x = m$, we get

$$\int_0^\infty \frac{p\sin pm}{p^2+1}\,dp = \frac{\pi}{2} e^{-|m|} = \frac{\pi}{2} e^{-m} \quad (\because m>0)$$

changing p to x, we get

$$\int_0^\infty \frac{x\sin mx}{x^2+1}\,dx = \frac{\pi}{2} e^{-m}$$

Example 6. Find the Fourier Sine transform of $\frac{e^{-ax}}{x}$

solution. Let $f(x) = \frac{e^{-ax}}{x}$

By the definition of Fourier sine transform,

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx = \int_0^\infty \frac{e^{-ax}}{x}\sin px\,dx$$

Let $I = \int_0^\infty \frac{e^{-ax}}{x}\sin px\,dx$ —— ①

Differentiating w.r.t. 'p' on both sides, we get

$$\frac{dI}{dp} = \frac{d}{dp}\left[\int_0^\infty \frac{e^{-ax}}{x}\sin px\,dx\right]$$

$$= \int_0^\infty \frac{e^{-ax}}{x}\cdot\frac{\partial}{\partial p}(\sin px)\,dx \qquad \left[\because \frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial}{\partial\alpha}\{f(x,\alpha)\}\,dx\right]$$

$$= \int_0^\infty \frac{e^{-ax}}{x}\cdot(x\cos px)\,dx$$

$$= \int_0^\infty e^{-ax}\cos px\,dx$$

$$= \left[\frac{e^{-ax}}{(-a)^2+p^2}\left(-a\cos px + p\sin px\right)\right]_{x=0}^{\infty}$$

$$= \left[0 - \frac{e^{-a(0)}}{a^2+p^2} (-a\cos 0 + p \sin 0) \right]$$

i.e., $\frac{dI}{dp} = \frac{a}{a^2+p^2}$

or $dI = \frac{a}{p^2+a^2} dp$

Integrating, we get

$$\int dI = a \int \frac{dp}{p^2+a^2} + c$$

i.e., $I = a \cdot \frac{1}{a} \tan^{-1}\left(\frac{p}{a}\right) + c$

$\therefore I = \tan^{-1}\left(\frac{p}{a}\right) + c$ ——— ②

From ① & ②, we obtain

$$\int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx = \tan^{-1}\left(\frac{p}{a}\right) + c$$

put $p = 0$, we get

$$\int_0^\infty \frac{e^{-ax}}{x} \sin 0 \, dx = \tan^{-1}\left(\frac{0}{a}\right) + c$$

i.e., $0 = \tan^{-1}(0) + c \Rightarrow c = 0$

Substituting c value in ②, we get

$$I = \tan^{-1}\left(\frac{p}{a}\right)$$

Hence $F_s \left\{ \frac{e^{-ax}}{x} \right\} = \tan^{-1}\left(\frac{p}{a}\right)$

Example 7. Show that (a) $F_s\{x f(x)\} = -\frac{d}{dp}[F_c(p)]$

(b) $F_c\{x f(x)\} = \frac{d}{dp}[F_s(p)]$

Solution.

(a) By the definition of Fourier cosine transform,

$$F_c(p) = F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx$$

Differentiating w.r.t. p on both sides, we get

$$\frac{d}{dp}[F_c(p)] = \frac{d}{dp}\left\{\int_0^\infty f(x)\cos px\,dx\right\}$$

$$= \int_0^\infty f(x)\cdot\frac{\partial}{\partial p}\{\cos px\}\,dx$$

$$= \int_0^\infty f(x)\cdot\{-x\sin px\}\,dx$$

$$= -\int_0^\infty \{x f(x)\}\cdot\sin px\,dx$$

i.e., $\frac{d}{dp}[F_c(p)] = -F_s\{x f(x)\}$ [$\because$ By definition of F.S.T]

or $F_s\{x f(x)\} = -\frac{d}{dp}[F_c(p)]$

(b) By the definition of Fourier sine transform,

$$F_s(p) = F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx$$

Differentiating w.r.t. p on both sides, we get

$$\frac{d}{dp}[F_s(p)] = \int_0^\infty f(x)\cdot\frac{\partial}{\partial p}\{\sin px\}\,dx$$

$$= \int_0^\infty f(x)\cdot\{x\cos px\}\,dx$$

$$= \int_0^\infty \{x f(x)\}\cos px\,dx$$

$= F_c\{x f(x)\}$ [$\because$ By definition of F.C.T]

$\therefore F_c\{x f(x)\} = \frac{d}{dp}[F_s(p)]$

Example 8. Find the Fourier cosine transform of e^{-x^2}. Hence evaluate Fourier sine transform of $x e^{-x^2}$

Solution. Let $f(x) = e^{-x^2}$

By the definition of Fourier cosine transform,

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx = \int_0^\infty e^{-x^2}\cos px\,dx$$

Let $I = \int_0^\infty e^{-x^2}\cos px\,dx$ —— (1)

Differentiating w.r.t. p on both sides, we get

$$\frac{dI}{dp} = \frac{d}{dp}\left\{\int_0^\infty e^{-x^2}\cos px\,dx\right\}$$

$$= \int_0^\infty e^{-x^2}\cdot\frac{\partial}{\partial p}\{\cos px\}\,dx$$

$$= \int_0^\infty e^{-x^2}\cdot\{-x\sin px\}\,dx$$

$$= \frac{1}{2}\int_0^\infty \underset{u}{\sin px}\cdot\{\underbrace{e^{-x^2}\cdot(-2x)}_{v}\}\,dx$$

$$= \frac{1}{2}\left[\sin px\int e^{-x^2}(-2x)\,dx - \int\left\{\frac{d}{dx}(\sin px)\int e^{-x^2}(-2x)\,dx\right\}dx\right]_0^\infty$$

$$= \frac{1}{2}\left[\underset{\text{bounded}}{\sin px}\cdot(e^{-x^2})\right]_0^\infty - \frac{1}{2}\int_0^\infty (p\cos px)\cdot(e^{-x^2})\,dx$$

$$= \frac{1}{2}[0] - \frac{p}{2}\int_0^\infty e^{-x^2}\cos px\,dx = -\frac{p}{2}I \quad [\because \text{By (1)}]$$

i.e., $\frac{dI}{dp} = -\frac{p}{2} I$

or $\frac{dI}{I} = -\frac{p}{2} dp$

Integrating, we get

$$\int \frac{dI}{I} = -\frac{1}{2}\int p\,dp + \log c$$

i.e., $\log I = -\frac{1}{2}\left(\frac{p^2}{2}\right) + \log c$

$\Rightarrow \log I - \log c = -\frac{p^2}{4}$

$\Rightarrow \log_e\left(\frac{I}{c}\right) = -\frac{p^2}{4}$

$\Rightarrow \frac{I}{c} = e^{-\frac{p^2}{4}}$

$\Rightarrow I = c\,e^{-\frac{p^2}{4}}$ —— (2)

From (1) & (2), we get

$$c\,e^{-\frac{p^2}{4}} = \int_0^\infty e^{-x^2}\cos px\,dx$$

put $p=0$, we get

$$c \cdot e^{-0} = \int_0^\infty e^{-x^2} \cdot \cos 0\,dx$$

i.e., $c = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

substituting c value in (2), we get

$$I = \frac{\sqrt{\pi}}{2}\,e^{-\frac{p^2}{4}}$$

Hence $F_c\{e^{-x^2}\} = \frac{\sqrt{\pi}}{2}\,e^{-\frac{p^2}{4}} = F_c(p)$, say

We know that $F_s\{x f(x)\} = -\frac{d}{dp}[F_c(p)]$

$$\therefore\ F_s\{x e^{-x^2}\} = -\frac{d}{dp}\left[\frac{\sqrt{\pi}}{2} e^{-\frac{p^2}{4}}\right]$$

$$= -\frac{\sqrt{\pi}}{2}\left[e^{-\frac{p^2}{4}}\left(-\frac{2p}{4}\right)\right]$$

$$= \frac{\sqrt{\pi}}{4} p e^{-\frac{p^2}{4}}$$

Example 9. Find the Fourier sine and cosine transforms of $x e^{-ax}$

Solution. Let $f(x) = e^{-ax}$

By the definition of Fourier sine transform

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin px \, dx$$

$$= \int_0^\infty e^{-ax} \sin px \, dx$$

$$= \left[\frac{e^{-ax}}{(-a)^2+p^2}\left[-a \sin px - p \cos px\right]\right]_{x=0}^{\infty}$$

$$= 0 - \frac{e^{-a(0)}}{a^2+p^2}(-a \sin 0 - p \cos 0)$$

i.e., $F_s\{e^{-ax}\} = \frac{p}{a^2+p^2} = F_s(p)$, say

By the definition of Fourier cosine transform

$$F_c\{f(x)\} = \int_0^\infty f(x) \cos px \, dx$$

$$= \int_0^\infty e^{-ax} \cos px \, dx$$

$$= \left[\frac{e^{-ax}}{(-a)^2+p^2}(-a \cos px + p \sin px)\right]_0^\infty$$

$$= 0 - \frac{e^{-a(0)}}{a^2+p^2}(-a\cos 0 + p\sin 0)$$

i.e., $F_c\{e^{-ax}\} = \frac{a}{a^2+p^2} = F_c(p)$, say

(i) We Know that $F_S\{x f(x)\} = -\frac{d}{dp}[F_c(p)]$

$$= -\frac{d}{dp}\left[\frac{a}{a^2+p^2}\right]$$

$$= -a\cdot\left[\frac{-1}{(a^2+p^2)^2}(2p)\right]$$

i.e., $F_S\{x e^{-ax}\} = \frac{2ap}{(a^2+p^2)^2}$

(ii) We Know that $F_c\{x f(x)\} = \frac{d}{dp}[F_S(p)]$

$$= \frac{d}{dp}\left[\frac{p}{a^2+p^2}\right]$$

$$= \frac{(a^2+p^2)(1) - p(2p)}{(a^2+p^2)^2}$$

i.e., $F_c\{x e^{-ax}\} = \frac{a^2-p^2}{(a^2+p^2)^2}$

Example 10. Find the Fourier cosine transform of $f(x) = \frac{1}{1+x^2}$. Hence derive Fourier sine transform of $\phi(x) = \frac{x}{1+x^2}$.

Solution. Given $f(x) = \frac{1}{1+x^2}$

By the definition of Fourier cosine transform,

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx$$

i.e., $F_c\left\{\frac{1}{1+x^2}\right\} = \int_0^\infty \frac{\cos px}{1+x^2}\,dx$

Let $I = \int_0^\infty \frac{\cos px}{1+x^2}\,dx$ —— (1)

Differentiating w.r.t. p on both sides, we get

$$\frac{dI}{dp} = \frac{d}{dp}\left\{\int_0^\infty \frac{\cos px}{1+x^2}\,dx\right\}$$

$$= \int_0^\infty \frac{1}{1+x^2} \cdot \frac{\partial}{\partial p}\{\cos px\,dx\}$$

$$= \int_0^\infty \frac{1}{1+x^2}(-x\sin px)\,dx$$

$$= -\int_0^\infty \frac{x}{1+x^2}\sin px\,dx$$

$$= -\int_0^\infty \frac{x^2}{x(1+x^2)}\sin px\,dx$$

$$= -\int_0^\infty \frac{(1+x^2)-1}{x(1+x^2)}\sin px\,dx$$

$$= -\int_0^\infty \left[\frac{(1+x^2)}{x(1+x^2)} - \frac{1}{x(1+x^2)}\right]\sin px\,dx$$

$$= -\int_0^\infty \left[\frac{1}{x} - \frac{1}{x(1+x^2)}\right]\sin px\,dx$$

$$= -\int_0^\infty \frac{\sin px}{x}\,dx + \int_0^\infty \frac{\sin px}{x(1+x^2)}\,dx$$

i.e., $\frac{dI}{dp} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin px}{x(1+x^2)}\,dx$ —— (2) $\left[\because \int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2},\ p>0\right]$

Again differentiating w.r.t. 'p' on both sides, we get

$$\frac{d^2I}{dp^2} = 0 + \int_0^\infty \frac{1}{x(1+x^2)}\frac{\partial}{\partial p}\{\sin px\}\,dx$$

$$= \int_0^\infty \frac{1}{\cancel{x}(1+x^2)} (\cancel{x} \cos px)\, dx$$

$$= \int_0^\infty \frac{\cos px}{1+x^2}\, dx$$

i.e., $\frac{d^2I}{dp^2} = I$ $\qquad [\because \text{By } ①]$

$\Rightarrow \frac{d^2I}{dp^2} - I = 0$

$\Rightarrow (D^2-1)I = 0$, where $D \equiv \frac{d}{dp}$

The A.E. is $m^2-1=0 \Rightarrow m = -1, 1$

$\therefore\ I = c_1 e^{-p} + c_2 e^{p}$ —— ③

Differentiating w.r.t. p on both sides, we get

$\frac{dI}{dp} = -c_1 e^{-p} + c_2 e^{p}$ —— ④

From ① & ③, we get $\int_0^\infty \frac{\cos px}{1+x^2} dx = c_1 e^{-p} + c_2 e^{p}$

put $p=0$, we get

$$\int_0^\infty \frac{\cos 0}{1+x^2} dx = c_1 e^{-0} + c_2 e^{0}$$

i.e., $\int_0^\infty \frac{1}{1+x^2} dx = c_1 + c_2$

$\Rightarrow \left[\tan^{-1}x\right]_0^\infty = c_1 + c_2$

$\Rightarrow \tan^{-1}(\infty) - \tan^{-1}(0) = c_1 + c_2$

$\Rightarrow c_1 + c_2 = \frac{\pi}{2}$ —— (i)

From ② & ④, we get $-\frac{\pi}{2} + \int_0^\infty \frac{\sin px}{x(1+x^2)} dx = -c_1 e^{-p} + c_2 e^{p}$

put $p=0$, we get

$$-\frac{\pi}{2} + \int_0^\infty \frac{\sin 0}{x(1+x^2)}\,dx = -c_1 e^{-0} + c_2 e^{0}$$

i.e., $-\frac{\pi}{2} + 0 = -c_1 + c_2 \Rightarrow c_1 - c_2 = \frac{\pi}{2}$ —(ii)

Solving (i) & (ii), we obtain $c_1 = \frac{\pi}{2}$ and $c_2 = 0$

From ③, $I = \frac{\pi}{2} e^{-p}$

Hence $F_c\left\{\frac{1}{1+x^2}\right\} = \frac{\pi}{2} e^{-p} = F_c(p)$ (say)

We know that $F_s\{x f(x)\} = -\frac{d}{dp}\left[F_c(p)\right]$

$$\text{i.e., } F_s\left\{\frac{x}{1+x^2}\right\} = -\frac{d}{dp}\left[\frac{\pi}{2} e^{-p}\right]$$

$$= -\frac{\pi}{2}\left[-e^{-p}\right]$$

$$= \frac{\pi}{2} e^{-p}$$

$$\therefore F_s\{\phi(x)\} = F_s\left\{\frac{x}{1+x^2}\right\} = \frac{\pi}{2} e^{-p}$$

Example 11. Using the Fourier sine transform of $e^{-ax} (a>0)$, show that $\int_0^\infty \frac{x \sin kx}{a^2+x^2}\,dx = \frac{\pi}{2} e^{-ak}$ $(k>0)$

Solution. Let $f(x) = e^{-ax}$

By the definition of Fourier sine transform,

$$F\{f(x)\} = \int_0^\infty f(x) \sin px\,dx$$

$$= \int_0^\infty e^{-ax} \sin px\,dx$$

$$= \frac{p}{a^2+p^2} = F_s(p), \text{ say} \quad [\text{Refer Example 9}]$$

By inversion formula for Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(p)\sin px\,dp$$

i.e., $$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\,dp$$

or $$\int_0^\infty \frac{p\sin px}{a^2+p^2}\,dp = \frac{\pi}{2}e^{-ax}$$

put $x = K$, we get

$$\int_0^\infty \frac{p\sin Kp}{a^2+p^2}\,dp = \frac{\pi}{2}e^{-aK}$$

changing p to x, we get

$$\int_0^\infty \frac{x\sin Kx}{a^2+x^2}\,dx = \frac{\pi}{2}e^{-aK} \quad (K>0$$

Example 12. Find the Fourier sine and cosine transform of

$$f(x) = \begin{cases} 1, & 0 \le x < 2 \\ 0, & x \ge 2 \end{cases}$$

solution. (i) By the definition of F.S.T,

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx$$

$$= \int_0^2 (1)\sin px\,dx + \int_2^\infty (0)\sin px\,dx$$

$$= -\left(\frac{\cos px}{p}\right)_0^2 + 0$$

$$= \frac{1-\cos 2p}{p}$$

(ii) By the definition of F.C.T,

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx = \frac{\sin 2p}{p}$$

Properties of Fourier transforms

(1) Linearity Property: If $F(p)$ and $G(p)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then $F\{af(x)+bg(x)\} = aF(p)+bG(p)$ where a and b are any constants

(2) Change of Scale property: If $F(p)$ is the Fourier transform of $f(x)$, then $F\{f(ax)\} = \frac{1}{a}F\left(\frac{p}{a}\right)$, $a \neq 0$

(3) Shifting property: If $F(p)$ is the Fourier transform of $f(x)$, then $F\{f(x-a)\} = e^{ipa}F(p)$

(4) Modulation theorem: If $F(p)$ is the Fourier transform of $f(x)$, then $F\{f(x)\cos ax\} = \frac{1}{2}\left[F(p+a)+F(p-a)\right]$

Example 13. Given $F\{e^{-x^2}\} = \sqrt{\pi}\, e^{-p^2/4}$, find the Fourier transform of (i) $e^{-x^2/3}$ (ii) $e^{-4(x-3)^2}$ (iii) $e^{-x^2}\cos 3x$

Solution. Let $f(x) = e^{-x^2}$ then $F\{f(x)\} = \sqrt{\pi}\, e^{-p^2/4} = F(p)$, (say)

(i) By change of scale property,

$$F\{f(ax)\} = \frac{1}{a}F\left(\frac{p}{a}\right)$$

Take $a = \frac{1}{\sqrt{3}}$,

$$F\left\{f\left(\frac{x}{\sqrt{3}}\right)\right\} = \frac{1}{\left(\frac{1}{\sqrt{3}}\right)} F\left(\frac{p}{1/\sqrt{3}}\right)$$

i.e., $$F\left\{e^{-(x/\sqrt{3})^2}\right\} = \sqrt{3}\, F(\sqrt{3}p) = \sqrt{3}\cdot\sqrt{\pi}\cdot e^{-(\sqrt{3}p)^2/4}$$

$$\therefore F\{e^{-x^2/3}\} = \sqrt{3\pi}\cdot e^{-3p^2/4}$$

(ii) By change of scale property,

$$F\{f(2x)\} = \frac{1}{2}F\left(\frac{p}{2}\right)$$

$$\text{i.e., } F\{e^{-(2x)^2}\} = \frac{1}{2}\sqrt{\pi}\cdot e^{-\left(\frac{p}{2}\right)^2/4}$$

$$\therefore F\{e^{-4x^2}\} = \frac{\sqrt{\pi}}{2}\cdot e^{-p^2/16}$$

Let $g(x) = e^{-4x^2}$ then $F\{g(x)\} = \frac{\sqrt{\pi}}{2}e^{-p^2/16} = G(p)$, say

By shifting property,

$$F\{g(x-a)\} = e^{ipa}G(p)$$

put $a = 3$,

$$F\{g(x-3)\} = e^{i3p}G(p)$$

$$\text{i.e., } F\{e^{-4(x-3)^2}\} = e^{3ip}\cdot\frac{\sqrt{\pi}}{2}e^{-p^2/16}$$

$$= \frac{\sqrt{\pi}}{2}\cdot e^{(3ip - p^2/16)}$$

(iii) By modulation theorem,

$$F\{f(x)\cos 3x\} = \frac{1}{2}\left[F(p+3) + F(p-3)\right]$$

$$\text{i.e., } F\{e^{-x^2}\cos 3x\} = \frac{1}{2}\left[\sqrt{\pi}\cdot e^{-(p+3)^2/4} + \sqrt{\pi}\,e^{-(p-3)^2/4}\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[e^{-(p+3)^2/4} + e^{-(p-3)^2/4}\right]$$

Example 14. Find the Fourier sine and cosine transform of x^{n-1} $(n>0)$. Hence deduce the Fourier sine and cosine transform of $\frac{1}{\sqrt{x}}$.

Solution. By the defn. of F.C.T,

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx$$

$$\text{i.e., } F_c\{x^{n-1}\} = \int_0^\infty x^{n-1}\cos px\,dx \quad \text{——— (1)}$$

By the defn. of F.S.T,

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx$$

$$\text{i.e., } F_s\{x^{n-1}\} = \int_0^\infty x^{n-1}\sin px\,dx \quad \text{——— (2)}$$

(1) + $i \times$ (2) gives

$$F_c\{x^{n-1}\} + iF_s\{x^{n-1}\} = \int_0^\infty x^{n-1}\cos px\,dx + i\int_0^\infty x^{n-1}\sin px\,dx$$

$$= \int_0^\infty x^{n-1}(\cos px + i\sin px)\,dx$$

$$= \int_0^\infty x^{n-1} e^{ipx}\,dx$$

put $ipx = -y$ i.e., $x = -\frac{y}{ip} = \frac{iy}{p}$

so that $dx = \frac{i}{p}dy$ $(\because \frac{1}{i} = -i)$

$$= \int_0^\infty \left(\frac{iy}{p}\right)^{n-1} \cdot e^{-y} \cdot \left(\frac{i}{p}\right) dy$$

$$= \int_0^\infty \left(\frac{i}{p}\right)^{n-1} \cdot y^{n-1} \cdot e^{-y} \cdot \left(\frac{i}{p}\right) dy$$

$$= \left(\frac{i}{p}\right)^{n-1+1} \int_0^\infty e^{-y}\, y^{n-1}\,dy$$

$$= \left(\frac{i}{p}\right)^n \cdot \Gamma(n) \qquad (\because \text{By the defn. of } \Gamma\text{-function})$$

$$= \frac{(i)^n}{p^n} \cdot \Gamma(n)$$

$$= \frac{\Gamma(n)}{p^n}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^n \qquad \left(\because i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= \frac{\Gamma(n)}{p^n}\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right) \qquad (\because \text{By De Moivre's theorem})$$

i.e., $F_c\{x^{n-1}\} + i F_s\{x^{n-1}\} = \frac{\Gamma(n)}{p^n}\cos\frac{n\pi}{2} + i\,\frac{\Gamma(n)}{p^n}\sin\frac{n\pi}{2}$

Equating real & imaginary parts, we get

$$F_c\{x^{n-1}\} = \frac{\Gamma(n)}{p^n}\cos\frac{n\pi}{2} \quad (3) \qquad \text{and} \qquad F_s\{x^{n-1}\} = \frac{\Gamma(n)}{p^n}\sin\frac{n\pi}{2} \quad (4)$$

Deduction: put $n = \frac{1}{2}$ in (3), we get

$$F_c\{x^{\frac{1}{2}-1}\} = \frac{\Gamma(\frac{1}{2})}{p^{1/2}}\cos\frac{\pi}{4}$$

$$= \frac{\sqrt{\pi}}{\sqrt{p}}\cdot\left(\frac{1}{\sqrt{2}}\right) \qquad \left(\because \Gamma(\tfrac{1}{2}) = \sqrt{\pi}\right)$$

i.e., $F_c\left\{\frac{1}{\sqrt{x}}\right\} = \sqrt{\frac{\pi}{2}}\cdot\frac{1}{\sqrt{p}}$

Similarly, putting $n = \frac{1}{2}$ in (4), we obtain

$$F_s\left\{\frac{1}{\sqrt{x}}\right\} = \sqrt{\frac{\pi}{2}}\cdot\frac{1}{\sqrt{p}}$$

$$\therefore F_s\left\{\frac{1}{\sqrt{x}}\right\} = F_c\left\{\frac{1}{\sqrt{x}}\right\} = \sqrt{\frac{\pi}{2}}\cdot\frac{1}{\sqrt{p}}$$

Example 15. If the Fourier sine transform of $f(x)$ is $\frac{e^{-ap}}{p}$, find $f(x)$.

Hence obtain the inverse Fourier sine transform of $\frac{1}{p}$.

Solution. Given $F_s(p) = \frac{e^{-ap}}{p}$

By inversion formula for F.S.T, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(p)\sin px\,dp$$

$$\text{i.e., } f(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-ap}}{p}\sin px\,dp \quad \text{——— (1)}$$

Differentiating w.r.t. x on both sides, we get

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty \frac{e^{-ap}}{p}\cdot\frac{\partial}{\partial x}\{\sin px\}\,dp$$

$$= \frac{2}{\pi}\int_0^\infty \frac{e^{-ap}}{\cancel{p}}\cdot(\cancel{p}\cos px)\,dp$$

$$= \frac{2}{\pi}\int_0^\infty e^{-ap}\cos px\,dp$$

$$= \frac{2}{\pi}\cdot\left[\frac{e^{-ap}}{(-a)^2+x^2}(-a\cos px + x\sin px)\right]_{p=0}^{\infty}$$

$$= \frac{2}{\pi}\left[0 - \frac{e^{-a(0)}}{a^2+x^2}(-a\cos 0 + x\sin 0)\right]$$

$$\text{i.e., } \frac{df}{dx} = \frac{2a}{\pi(a^2+x^2)}$$

$$\text{or} \quad df = \frac{2a}{\pi(a^2+x^2)}dx$$

Integrating, we get

$$\int df = \frac{2a}{\pi}\cdot\int\frac{dx}{x^2+a^2}$$

$$\text{i.e., } f(x) = \frac{2a}{\pi}\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

or $\quad f(x) = \frac{2}{\pi} \tan^{-1}\left(\frac{x}{a}\right) + c$ —— ②

From ① & ②, we have

$$\frac{2}{\pi} \int_0^\infty \frac{e^{-ap}}{p} \sin px \, dp = \frac{2}{\pi} \tan^{-1}\left(\frac{x}{a}\right) + c$$

put $x = 0$, we get

$$\frac{2}{\pi} \int_0^\infty \frac{e^{-ap}}{p} (\sin 0) \, dp = \frac{2}{\pi} \tan^{-1}(0) + c$$

i.e., $\frac{2}{\pi}(0) = \frac{2}{\pi}(0) + c \Rightarrow c = 0$

From ②, $\quad f(x) = \frac{2}{\pi} \tan^{-1}\left(\frac{x}{a}\right)$

Since $f(x) = F_s^{-1}\{F_s(p)\}$, we have

$$F_s^{-1}\left\{\frac{e^{-ap}}{p}\right\} = \frac{2}{\pi} \tan^{-1}\left(\frac{x}{a}\right) \text{ —— ③}$$

put $a = 0$ in ③, we get

$$F_s^{-1}\left\{\frac{e^{-a(0)}}{p}\right\} = \frac{2}{\pi} \tan^{-1}\left(\frac{x}{0}\right) = \frac{2}{\pi} \tan^{-1}(\infty) = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1$$

$$\therefore \; F_s^{-1}\left\{\frac{1}{p}\right\} = 1$$

Example 16. Obtain Fourier sine transform of $f(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4 - x, & 1 < x < 4 \\ 0, & x > 4 \end{cases}$

solution. By the defn. of F.S.T, we have

$$F\{f(x)\} = \int_0^\infty f(x) \sin px \, dx$$

$$= \int_0^1 (4x) \sin px \, dx + \int_1^4 (4-x) \sin px \, dx + \int_4^\infty (0) \sin px \, dx$$

$$= \frac{1}{p^2}(5 \sin p - p \cos p - \sin 4p)$$

Stoke's theorem (Relation between line and surface integrals)

Statement: If S is an open surface bounded by a closed curve C and $\vec{F}$ be any continuously differentiable vector point function, then $\oint_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds$, where $\vec{n}$ is unit outward normal at any point of S.

Note: Green's theorem in a plane is a special case of Stoke's theorem.

Example 1. Apply Stoke's theorem to evaluate $\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$ where C is the boundary of the rectangle $0 \le x \le \pi,\ 0 \le y \le 1,\ z = 3$.

Solution: By Stoke's theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds$$

i.e., $$\oint_C (F_1\,dx + F_2\,dy + F_3\,dz) = \int_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds \quad \text{——} (1)$$

Here $F_1 = \sin z$, $F_2 = -\cos x$, $F_3 = \sin y$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}$$

$$= \vec{i}(\cos y - 0) - \vec{j}(0 - \cos z) + \vec{k}(\sin x - 0)$$

$$= \cos y\,\vec{i} + \cos z\,\vec{j} + \sin x\,\vec{k}$$

Let R be the projection of S in xy-plane. Then $\vec{n} = \vec{k}$ and $ds = dx\,dy$ ($\because$ the rectangle lies in xy-plane)

$\therefore$ curl $\vec{F}\cdot\vec{n} = (\cos y\,\vec{i} + \cos z\,\vec{j} + \sin x\,\vec{k})\cdot\vec{k} = \sin x\,(\vec{k}\cdot\vec{k}) = \sin x$

$(\because \vec{k}\cdot\vec{k}=1)$

From ①, we have

$$\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz) = \int_S \sin x\,ds$$

$$= \iint_R \sin x\,dx\,dy \quad (\because ds = dx\,dy)$$

$$= \int_{y=0}^{1}\int_{x=0}^{\pi} \sin x\,dx\,dy$$

$$= \left(\int_{y=0}^{1} dy\right)\left(\int_{x=0}^{\pi} \sin x\,dx\right)$$

$$= (y)_0^1\,(-\cos x)_0^{\pi}$$

$$= (1-0)(-\cos\pi + \cos 0)$$

$$= 2$$

Example 2. If $\vec{F} = 3y\,\vec{i} - xz\,\vec{j} + yz^2\,\vec{k}$ and S is the surface of the paraboloid $2z = x^2+y^2$ bounded by $z=2$, evaluate $\iint_S (\nabla\times\vec{F})\cdot d\vec{S}$ using Stoke's theorem.

Solution: By Stoke's theorem,

$$\int_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds = \oint_C \vec{F}\cdot d\vec{r}$$

i.e., $$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} \quad \text{——} ①$$

where C is the circle $x^2+y^2=4$ $(\because z=2)$

$$\therefore \vec{F}\cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz$$

$$= 3y\,dx - xz\,dy + yz^2 dz$$

$$= 3y\,dx - 2x\,dy \qquad (\because z = 2 \Rightarrow dz = 0)$$

From ①, we have

$$\iint_S (\nabla \times \vec{F})\cdot d\vec{S} = \oint_C (3y\,dx - 2x\,dy)$$

The parametric equations of C are $x = 2\cos t$, $y = 2\sin t$

so that $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$, where $0 \le t \le 2\pi$

$$= \int_{t=0}^{2\pi} [3(2\sin t)(-2\sin t)dt - 2(2\cos t)(2\cos t)dt]$$

$$= \int_0^{2\pi} (-12\sin^2 t - 8\cos^2 t)\,dt$$

$$= \int_0^{2\pi} (-4\sin^2 t)\,dt - 8\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= -2\int_0^{2\pi} (2\sin^2 t)\,dt - 8\int_0^{2\pi} dt$$

$$= -2\int_0^{2\pi} (1 - \cos 2t)\,dt - 8(t)_0^{2\pi}$$

$$= -2\left(t - \frac{\sin 2t}{2}\right)_0^{2\pi} - 8(2\pi)$$

$$= -2\left(2\pi - \frac{\sin 4\pi}{2} - 0\right) - 16\pi$$

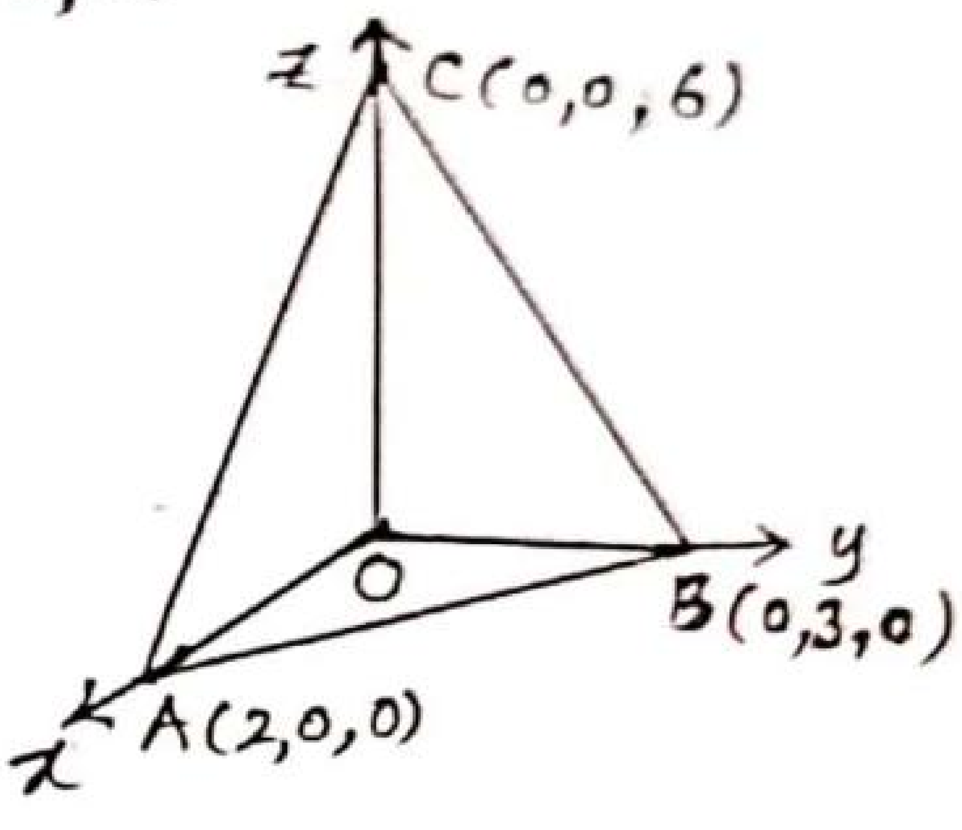
$$= -4\pi - 16\pi$$

$$= -20\pi$$

Hence $\iint_S (\nabla \times \vec{F})\cdot d\vec{S} = -20\pi$

Example 3. Use Stoke's theorem to evaluate

$\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$, where C is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$.



solution. By Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds$$

i.e., $$\int_C (F_1dx + F_2dy + F_3dz) = \int_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds \quad \text{——} (1)$$

Here $F_1 = (x+y)$; $F_2 = (2x-z)$; $F_3 = (y+z)$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y) & (2x-z) & (y+z) \end{vmatrix}$$

$$= \vec{i}(1+1) - \vec{j}(0-0) + \vec{k}(2-1)$$

$$= 2\vec{i} + \vec{k}$$

Equation of the plane passing through A, B, C is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \quad \text{i.e.,} \quad 3x+2y+z = 6.$$

Let $\phi \equiv 3x+2y+z-6 = 0$ be the given surface

The normal vector to the surface is $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

$$= 3\vec{i} + 2\vec{j} + \vec{k}$$

$\therefore$ Unit outward normal vector $\vec{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{1}{\sqrt{14}}(3\vec{i} + 2\vec{j} + \vec{k})$$

From ①, we have

$$\int_C [(x+y)dx + (2x-z)dy + (y+z)dz] = \int_S (2\vec{i}+\vec{k})\cdot\frac{1}{\sqrt{14}}(3\vec{i}+2\vec{j}+\vec{k})ds$$

$$= \frac{1}{\sqrt{14}}\int_S (6+1)\,ds$$

$$= \frac{7}{\sqrt{14}}\int_S ds$$

$$= \frac{7}{\sqrt{14}}(\text{Area of } \Delta ABC)$$

$$= \frac{7}{\sqrt{14}} \times \frac{1}{2}|(\vec{AB}\times\vec{AC})| \quad \text{—— ②}$$

$\vec{AB} = \vec{OB} - \vec{OA} = 3\vec{j} - 2\vec{i}$ and $\vec{AC} = \vec{OC} - \vec{OA} = 6\vec{k} - 2\vec{i}$

$$\therefore \vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix} = \vec{i}(18-0) - \vec{j}(-12-0) + \vec{k}(0+6)$$

$$= 18\vec{i} + 12\vec{j} + 6\vec{k}$$

$$= 6(3\vec{i} + 2\vec{j} + \vec{k})$$

$$|(\vec{AB}\times\vec{AC})| = |6(3\vec{i}+2\vec{j}+\vec{k})| = 6\sqrt{9+4+1} = 6\sqrt{14} \quad \text{—— ③}$$

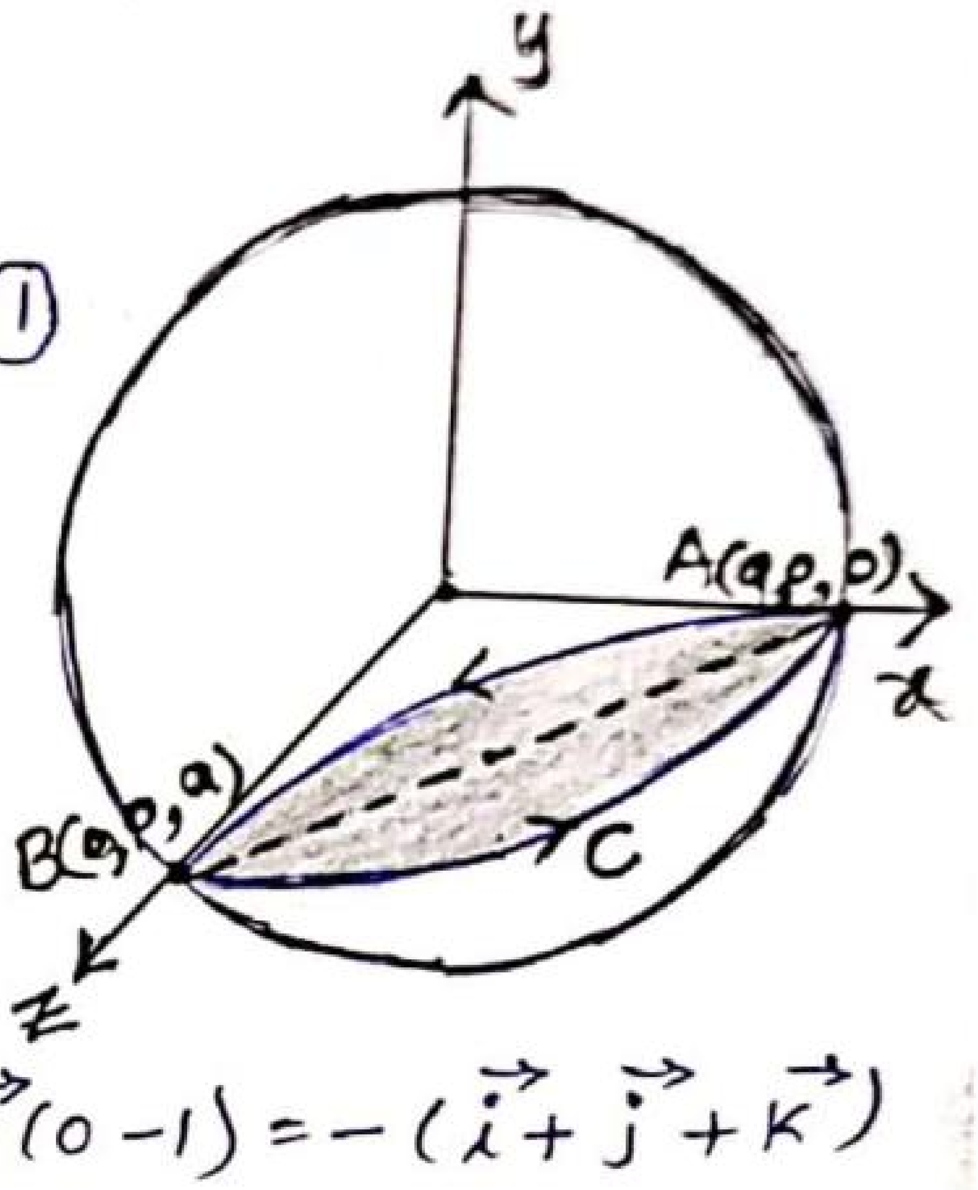
From ② & ③, we get

$$\int_C [(x+y)dx + (2x-z)dy + (y+z)dz] = \frac{7}{\sqrt{14}} \times \frac{1}{2}(6\sqrt{14}) = 21$$

Example 4. Apply Stoke's theorem to evaluate $\int_C (y\,dx + z\,dy + x\,dz)$, where C is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$.

Solution. By Stoke's theorem,

$$\int_C (F_1 dx + F_2 dy + F_3 dz) = \int_S \operatorname{curl}\vec{F}\cdot\vec{n}\,ds \quad \text{—} \; (1)$$



Here $F_1 = y$; $F_2 = z$; $F_3 = x$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \vec{i}(0-1) - \vec{j}(1-0) + \vec{k}(0-1) = -(\vec{i}+\vec{j}+\vec{k})$$

The curve C is a circle in the plane $x+z=a$ having diameter $AB = \sqrt{a^2+0+a^2} = a\sqrt{2}$

Let $\phi \equiv x+z-a=0$ be the given surface

The normal vector to the surface is $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

$$= \vec{i}(1) + \vec{j}(0) + \vec{k}(1)$$

$$= \vec{i} + \vec{k}$$

$\therefore$ Unit outward normal vector $\vec{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{1}{\sqrt{2}}(\vec{i}+\vec{k})$

From (1), we have

$$\int_C (y\,dx + z\,dy + x\,dz) = \int_S -(\vec{i}+\vec{j}+\vec{k})\cdot\frac{1}{\sqrt{2}}(\vec{i}+\vec{k})\,ds$$

$$= -\frac{1}{\sqrt{2}} \int_S (1+1)\, ds$$

$$= -\frac{2}{\sqrt{2}} \int_S ds$$

$$= -\sqrt{2}\,(A), \text{ where } A \text{ is the area of circle } C.$$

$$= -\sqrt{2}\cdot \pi\left(\frac{a}{\sqrt{2}}\right)^2 \qquad \left[\because \text{radius} = \tfrac{1}{2}(a\sqrt{2}) = \tfrac{a}{\sqrt{2}}\right]$$

$$= -\frac{\pi a^2}{\sqrt{2}}$$

Hence $\int_C (y\,dx + z\,dy + x\,dz) = -\frac{\pi}{\sqrt{2}} a^2$

Example 5. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = y\vec{i} + xz^3\vec{j} - zy^3\vec{k}$ and C is the circle $x^2+y^2=4$, $z=1.5$.

Solution: By Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\vec{n}\, ds \quad \text{——} \ (1)$$

Given $\vec{F} = y\vec{i} + xz^3\vec{j} - zy^3\vec{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz^3 & -zy^3 \end{vmatrix} = -3(y^2 z + xz^2)\vec{i} + (z^3-1)\vec{k}$$

Let R be the projection of S on xy-plane. Then

$\vec{n} = \vec{k}$ and $ds = dx\,dy$

From ①, we have

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \left[-3(y^2z + xz^2)\vec{i} + (z^3-1)\vec{k}\right]\cdot\vec{k}\,ds$$

$$= \iint_R (z^3-1)\,dx\,dy$$

$$= \iint_R \left[(1.5)^3-1\right]dx\,dy \quad [\because z = 1.5]$$

$$= \frac{19}{8}\iint_R dx\,dy$$

$= \frac{19}{8}(A)$, where A is the area of the circle $C: x^2+y^2=4$

$$= \frac{19}{8}\times \pi(2)^2 = \frac{19}{2}\pi$$

Hence $\int_C \vec{F}\cdot d\vec{r} = \frac{19}{2}\pi$

Example 6. Use Stoke's theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ and C is the rectangle in the xy-plane bounded by $y=0$, $x=a$, $y=b$ and $x=0$. [Ans: $-2ab^2$]

[Hint: Refer Example 1]

Example 7. Evaluate by Stoke's theorem $\oint_C (yz\,dx + zx\,dy + xy\,dz)$, where C is the curve $x^2+y^2=1$, $z=y^2$. [Ans: 0]

Sol: By Stoke's theorem,

$$\oint_C (F_1dx + F_2dy + F_3dz) = \int_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds = 0 \quad (\because \text{curl}\,\vec{F} = \vec{0})$$

Gauss's divergence theorem (Relation between surface and volume integrals)

Statement: If S is a closed surface enclosing a volume V and $\vec{F}$ be any continuously differentiable vector point function,

then $\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \operatorname{div}\vec{F}\,dv$ or $\int_S \vec{F}\cdot d\vec{s} = \int_V \operatorname{div}\vec{F}\,dv$

Gauss's divergence theorem in cartesian form:

$$\iint_S (F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\,dy\,dz$$

Example 1. Use divergence theorem to evaluate $\int_S \vec{F}\cdot d\vec{s}$, where $\vec{F} = [e^x, e^y, e^z]$ and S is the surface of the cube $|x|\le 1$, $|y|\le 1$, $|z|\le 1$.

Solution. By divergence theorem,

$$\int_S \vec{F}\cdot d\vec{s} = \int_V \operatorname{div}\vec{F}\,dV \quad \text{——— (1)}$$

Given $\vec{F} = [e^x, e^y, e^z] = e^x\vec{i} + e^y\vec{j} + e^z\vec{k}$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(e^y) + \frac{\partial}{\partial z}(e^z) = e^x + e^y + e^z$$

From (1), $\int_S \vec{F}\cdot d\vec{s} = \int_V (e^x + e^y + e^z)\,dv$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (e^x + e^y + e^z)\,dx\,dy\,dz \qquad (\because dv = dx\,dy\,dz)$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^{x}\,dx\,dy\,dz + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^{y}\,dx\,dy\,dz + \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^{z}\,dx\,dy\,dz$$

$$= \left(\int_{-1}^{1} e^{x}dx\right)\left(\int_{-1}^{1} dy\right)\left(\int_{-1}^{1} dz\right) + \left(\int_{-1}^{1} dx\right)\left(\int_{-1}^{1} e^{y}dy\right)\left(\int_{-1}^{1} dz\right) + \left(\int_{-1}^{1} dx\right)\left(\int_{-1}^{1} dy\right)\left(\int_{-1}^{1} e^{z}dz\right)$$

$$= (e^{x})_{-1}^{1}(y)_{-1}^{1}(z)_{-1}^{1} + (x)_{-1}^{1}(e^{y})_{-1}^{1}(z)_{-1}^{1} + (x)_{-1}^{1}(y)_{-1}^{1}(e^{z})_{-1}^{1}$$

$$= (e-e^{-1})(1+1)(1+1) + (1+1)(e-e^{-1})(1+1) + (1+1)(1+1)(e-e^{-1})$$

$$= 4(e-e^{-1}) + 4(e-e^{-1}) + 4(e-e^{-1})$$

$$= 12(e-e^{-1})$$

Example 2. Evaluate $\int_S (a^2x^2+b^2y^2+c^2z^2)^{-1/2}\,ds$, where S is the surface of the ellipsoid $ax^2+by^2+cz^2=1$.

Solution. Let $\phi \equiv ax^2+by^2+cz^2=1 \rightarrow (1)$ be the given surface

we have $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

$$= \vec{i}(2ax) + \vec{j}(2by) + \vec{k}(2cz)$$

$$= 2(ax\vec{i} + by\vec{j} + cz\vec{k})$$

$$\therefore \vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\cancel{2}(ax\vec{i}+by\vec{j}+cz\vec{k})}{\cancel{2}\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \frac{ax\vec{i}+by\vec{j}+cz\vec{k}}{(a^2x^2+b^2y^2+c^2z^2)^{1/2}}$$

Given $\vec{F}\cdot\vec{n} = (a^2x^2+b^2y^2+c^2z^2)^{-1/2}$

i.e., $\vec{F}\cdot\dfrac{(ax\vec{i}+by\vec{j}+cz\vec{k})}{\cancel{(a^2x^2+b^2y^2+c^2z^2)^{1/2}}} = \dfrac{1}{\cancel{(a^2x^2+b^2y^2+c^2z^2)^{1/2}}}$

$$\Rightarrow \vec{F}\cdot(ax\vec{i}+by\vec{j}+cz\vec{k}) = ax^2+by^2+cz^2 \quad [\because \text{By } ①,\ ax^2+by^2+cz^2=1]$$

$$\Rightarrow \vec{F}\cdot(ax\vec{i}+by\vec{j}+cz\vec{k}) = (x\vec{i}+y\vec{j}+z\vec{k})\cdot(ax\vec{i}+by\vec{j}+cz\vec{k})$$

$$\Rightarrow \vec{F} = x\vec{i}+y\vec{j}+z\vec{k}$$

By divergence theorem, we have

$$\int_S \vec{F}\cdot\vec{n}\, ds = \int_V \operatorname{div}\vec{F}\, dv$$

i.e., $$\int_S (a^2x^2+b^2y^2+c^2z^2)^{-1/2} ds = \int_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dv$$

$$= \int_V \left[\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)\right] dv$$

$$= \int_V (1+1+1)\, dv$$

$$= 3\int_V dv$$

$= 3(V)$, where V is the volume of the ellipsoid $ax^2+by^2+cz^2=1$ i.e., $\frac{x^2}{(\frac{1}{\sqrt{a}})^2}+\frac{y^2}{(\frac{1}{\sqrt{b}})^2}+\frac{z^2}{(\frac{1}{\sqrt{c}})^2}=1$

$$= 3\cdot\frac{4\pi}{3}\left(\frac{1}{\sqrt{a}}\right)\left(\frac{1}{\sqrt{b}}\right)\left(\frac{1}{\sqrt{c}}\right)$$

$$= \frac{4\pi}{\sqrt{abc}} \quad \left[\because \text{Volume of } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = \frac{4\pi}{3}abc\right]$$

Example 3. Evaluate $\int_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = 4x\vec{i}-2y^2\vec{j}+z^2\vec{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$.

solution. By divergence theorem,

$$\int_S \vec{F}\cdot d\vec{S} = \int_V \operatorname{div}\vec{F}\, dv \quad \text{——} ①$$

Given $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

$$\therefore \operatorname{div}\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$$

From ①, we have

$$\int_S \vec{F}\cdot d\vec{S} = \int_V (4-4y+2z)\,dv$$

$$x^2+y^2=4 \Rightarrow y = \pm\sqrt{4-x^2}$$

put $y=0$, we get $x^2=4 \Rightarrow x = \pm 2$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{3}(4-4y+2z)\,dx\,dy\,dz \qquad (\because dv = dx\,dy\,dz)$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[\int_{z=0}^{3}(4-4y+2z)\,dz\right]dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[(4-4y)z + 2\left(\frac{z^2}{2}\right)\right]_{z=0}^{3}dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[(4-4y)(3) + (3)^2 - 0\right]dy\,dx$$

$$= \int_{x=-2}^{2}\left\{\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)\,dy\right\}dx$$

$$= \int_{x=-2}^{2}\left\{21y - 12\left(\frac{y^2}{2}\right)\right\}_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

$$= \int_{-2}^{2}\left\{21(\sqrt{4-x^2}) - \cancel{6(4-x^2)} + 21(\sqrt{4-x^2}) + \cancel{6(4-x^2)}\right\}dx$$

$$= 42\int_{-2}^{2}\underbrace{\sqrt{4-x^2}}_{\text{even fn}}\,dx = 42\cdot 2\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{x}{2}\sqrt{2^2-x^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 84\left[0 + 2\sin^{-1}\left(\frac{2}{2}\right) - 0\right]$$

$$= 84\left[2\left(\frac{\pi}{2}\right) = 84\pi\right.$$

Example 4. By transforming to triple integral, evaluate $\iint_S (x^3 dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy)$, where S is the closed surface consisting of the cylinder $x^2+y^2=a^2$ and the circular discs $z=0$ and $z=b$.

Solution. By divergence theorem,

$$\iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz.$$

i.e., $$\iint_S (x^3 dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy) = \iiint_V \left[\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(x^2z)\right] dx\,dy\,dz$$

$$= \iiint_V (3x^2 + x^2 + x^2)\,dx\,dy\,dz$$

$$= 5\iiint_V x^2\,dx\,dy\,dz$$

$$x^2+y^2=a^2 \Rightarrow y = \pm\sqrt{a^2-x^2}$$

put $y=0$, we get $x^2=a^2 \Rightarrow x = \pm a$

$$= 5\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2+x^2}}\int_{z=0}^{b} x^2\,dz\,dy\,dx$$

$$= 5 \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left\{ \int_{z=0}^{b} x^2\, dz \right\} dy\, dx$$

$$= 5 \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left\{ x^2 (z) \right\}_{z=0}^{b} dy\, dx$$

$$= 5 \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2 (b-0)\, dy\, dx$$

$$= 5b \int_{-a}^{a} \left\{ \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\, dy \right\} dx$$

$$= 5b \int_{-a}^{a} \left\{ x^2 (y) \right\}_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= 5b \int_{-a}^{a} \left\{ x^2 \left(\sqrt{a^2-x^2} + \sqrt{a^2-x^2}\right) \right\} dx$$

$$= 10b \int_{-a}^{a} x^2 \sqrt{a^2-x^2}\, dx$$

$$= 10b \cdot 2 \int_{0}^{a} x^2 \sqrt{a^2-x^2}\, dx \qquad (\because \text{Integrand is even function})$$

put $x = a\sin\theta$ so that $dx = a\cos\theta\, d\theta$

$$= 20b \int_{0}^{\pi/2} (a\sin\theta)^2 \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta\, d\theta$$

$$= 20b \int_{0}^{\pi/2} a^2 \sin^2\theta \cdot \sqrt{a^2(1-\sin^2\theta)} \cdot a\cos\theta\, d\theta$$

$$= 20a^4 b \int_{0}^{\pi/2} \sin^2\theta \cdot \cos^2\theta\, d\theta$$

$$= 20a^4 b \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5}{4}\pi a^4 b$$

Hence $\iint_S (x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy) = \frac{5}{4}\pi a^4 b$

Example 5. Using divergence theorem, evaluate $\int_S \vec{r}\cdot\vec{n}\,ds$, where S is the surface of the sphere $x^2+y^2+z^2=9$.

Solution. By divergence theorem,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \operatorname{div}\vec{F}\,dv$$

i.e., $$\int_S \vec{r}\cdot\vec{n}\,ds = \int_V \operatorname{div}\vec{r}\,dv$$

$$= \int_V (3)\,dv \qquad \left(\because \vec{r} = x\vec{i}+y\vec{j}+z\vec{k} \Rightarrow \operatorname{div}\vec{r} = 3\right)$$

$$= 3\int_V dv$$

$= 3(V)$, where V is the volume of the sphere $x^2+y^2+z^2=3^2$

$$= \cancel{3}\cdot\frac{4\pi}{\cancel{3}}(3)^3$$

$$= 108\pi$$

Example 6. If S is any closed surface enclosing a volume V and $\vec{F} = ax\vec{i}+by\vec{j}+cz\vec{k}$, prove that $\int_S \vec{F}\cdot\vec{n}\,ds = (a+b+c)V$.

Solution. $$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz) = a+b+c$$

By divergence theorem,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \operatorname{div}\vec{F}\,dv$$

$$= \int_V (a+b+c)\,dv$$

$= (a+b+c)\int_V dv = (a+b+c)V$, where V is the volume of the surface S